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Non-commutative symplectic $\mathbb{N}$Q-geometry and Courant algebroids

Memoria de Tesis Doctoral presentada por

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A mis Padres

That long black cloud is comin' down,
I feel like I'm knockin' on heaven's door.

Knock, knock, knockin' on heaven's door,
Knock, knock, knockin' on heaven's door.

Bob Dylan

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No han sido años fáciles. Supongo que nunca los son. He conocido el sabor del desánimo, del desaliento, de la incertidumbre, casi de la desesperación. También he saboreado la emoción por el descubrimiento, por ir un paso más allá, en definitiva, por sacar lo mejor de mí mismo. Quizás haya aprendido a hacerme mayor; lo que seguro que he aprendido es que la vida iba en serio. Creo que en algunos aspectos he ganado como persona; pero me temo haber perdido en otros. Espero que el balance salga positivo. Pero, sobre todo, estos años se han convertido en un camino de aprendizaje y éste es el momento de recordar a las personas que me han dado tanto y me han permitido completarlo.

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Abstract

We propose a notion of non-commutative Courant algebroid that satisfies the Kontsevich–Rosenberg principle, whereby a structure on an associative algebra has geometric meaning if it induces standard geometric structures on its representation spaces. Replacing vector fields on varieties by Crawley-Boevey's double derivations on associative algebras, this principle has been successfully applied by Crawley-Boevey, Etingof and Ginzburg to symplectic structures and by Van den Bergh to Poisson structures.

Courant algebroids, introduced in differential geometry by Liu, Weinstein and Xu, generalize the notion of the Drinfeld double to Lie bialgebroids. They axiomatize the properties of the Courant bracket, introduced by Courant and Weinstein to provide a geometric setting for Dirac's theory of constrained mechanical systems. A direct approach to define non-commutative Courant algebroids fails, because the Cartan identities are unknown in the calculus of non-commutative differential forms and double derivations, so in this thesis we follow an indirect method.

Symplectic $\mathbb{N}$Q-manifolds are non-negatively graded manifolds (the grading is called weight), endowed with a graded symplectic structure and a symplectic homological vector field Q of weight 1. They encode higher Lie algebroid structures in the Batalin–Vilkovisky formalism in physics, where the weight keeps track of the ghost number. Following ideas of Ševera, Roytenberg proved that symplectic $\mathbb{N}$Q-manifolds of weights 1 and 2 are in 1-1 correspondence with Poisson manifolds and Courant algebroids, respectively. Our method to construct non-commutative Courant algebroids is to adapt this result to a graded version of the formalism of Crawley-Boevey, Etingof and Ginzburg.

We start generalizing to graded associative algebras the theories of bi-symplectic forms and double Poisson brackets of Crawley-Boevey–Etingof–Ginzburg and Van den Bergh, respectively. In this framework, we prove suitable Darboux theorems for graded bi-symplectic forms, define bi-symplectic $\mathbb{N}$Q-algebras, and prove a 1-1 correspondence between appropriate bi-symplectic $\mathbb{N}$Q-algebras of weight 1 and Van den Berg's double Poisson algebras. We then use suitable non-commutative Lie and Atiyah algebroids to describe bi-symplectic $\mathbb{N}$-graded algebras of weight 2 whose underlying graded algebras are graded-quiver path algebras, in terms Van den Berg's pairings on projective bimodules. Using non-commutative derived brackets, we calculate the algebraic structure that corresponds to symplectic $\mathbb{N}$Q-algebras of this type. By the analogy with Roytenberg's correspondence for commutative algebras, we call this structure a double Courant–Dorfman algebra.

Resumen

En esta tesis proponemos una noción de algebroide de Courant no conmutativo que satisface el principio de Kontsevich–Rosenberg, según el cual una estructura sobre un álgebra asociativa tiene significado geométrico si induce las estructuras geométricas estándar sobre sus espacios de representaciones. Reemplazando los campos vectoriales sobre variedades por las derivaciones dobles de Crawley-Boevey sobre álgebras asociativas, este principio ha sido aplicado con éxito por Crawley-Boevey, Etingof y Ginzburg para estructuras simplécticas, y por Van den Bergh para estructuras de Poisson.

Los algebroides de Courant, introducidos por Liu, Weinstein y Xu, generalizan la noción de doble de Drinfeld a bialgebroides de Lie, y axiomatizan las propiedades del corchete de Courant definido por Courant y Weinstein para dotar de un contexto geométrico a la teoría de Dirac de sistemas mecánicos con ligaduras. Un enfoque directo para definir algebroides de Courant no es posible porque las identidades de Cartan no se conocen en el cálculo de formas diferenciales no conmutativas y derivaciones dobles, así que en esta tesis seguimos un método indirecto.

Las $\mathbb{N}Q$-variedades simplécticas son variedades graduadas no negativamente (la graduación se llama peso), dotadas con una estructura simpléctica graduada y un campo vectorial homológico Q de peso 1. Estas estructuras codifican estructuras de algebroide de Lie de orden superior en el formalismo de Batalin–Vilkovisky en Física, donde los pesos tienen en cuenta el número fantasma. Siguiendo ideas de Ševera, Roytenberg probó que las $\mathbb{N}Q$-variedades simplécticas de pesos 1 y 2 están en correspondencia 1-1 con variedades de Poisson y algebroides de Courant, respectivamente. Nuestro método para construir algebroides de Courant no conmutativos consiste en adaptar este resultado a una versión graduada del formalismo de Crawley-Boevey, Etingof, Ginzburg.

Empezamos generalizando a álgebras asociativas graduadas las teorías de formas bi-simplécticas y corchetes dobles de Poisson de Crawley-Boevey–Etingof–Ginzburg y Van den Bergh, respectivamente. En este contexto, probamos teoremas de Darboux adecuados para formas bi-simplécticas, definimos $\mathbb{N}Q$-álgebras bi-simplécticas, y probamos una correspondencia 1-1 entre $\mathbb{N}Q$-álgebras bi-simplécticas apropiadas de peso 1 y álgebras de Poisson dobles de Van den Bergh. Entonces usamos algebroides de Lie y de Atiyah adecuados para describir álgebras $\mathbb{N}$-graduadas de peso 2 cuyas álgebras graduadas subyacentes son álgebras de caminos de carcajs graduados, en términos de emparejamientos de Van den

Bergh sobre bimódulos proyectivos. Usando corchetes derivados no conmutativos, calculamos la estructura algebraica que corresponde a $\mathbb{N}Q$-álgebras bi-simplécticas de este tipo. Por analogía con la correspondencia de Roytenberg para álgebras conmutativas, llamaremos a esta estructura un álgebra de Courant–Dorfman doble.

Contents

Chapter 1

Introduction

This thesis can be framed into a program to define geometric structures on non-commutative algebras. More precisely, the main aim is to define a structure on an associative algebra that induces a structure of Courant algebroid on its representation schemes in finite-dimensional vector spaces. Following the Kontsevich–Rosenberg principle that we will now review, these structures will be called non-commutative Courant algebroids.

1.1 Geometric structures on representation spaces

A general approach, used since the 1970s, to study the representation theory of a (unital) finitely generated associative algebra A over a field k consists in studying the geometry of its representation schemes (see [15, 23]). By definition, the representation scheme $\operatorname{Rep}(A,V)$ of A in a finite-dimensional vector space V is the affine scheme representing the functor from the category $\mathtt{CommAlg}_k$ of (unital) finitely generated commutative k-algebras into the category $\mathtt{Sets}$ of sets, given by

$$\operatorname{Rep}(A,V)^\natural\colon \mathtt{CommAlg}_k \longrightarrow \mathtt{Sets}\colon \quad B \longmapsto \operatorname{Hom}_{\mathtt{Alg}_k}(A, \operatorname{End} V \otimes B).$$

The fact that this functor is representable means that there exists an affine scheme

$$\operatorname{Rep}(A,V) = \operatorname{Spec}(A_V),$$

for a finitely generated commutative k-algebra A_V and isomorphisms

$$\operatorname{Hom}_{\mathtt{CommAlg}_k}(A_V, B) \simeq \operatorname{Hom}_{\mathtt{Alg}_k}(A, \operatorname{End} V \otimes B),$$

natural in $B \in \mathtt{CommAlg}_k$. A simple way to construct $\operatorname{Rep}(A,V)$ is to define its coordinate ring A_V as the commutative algebra with set of generators $\{a_{jl} \mid a \in A,\ 1 \leq j,l \leq N\}$, for a fixed isomorphism $V \cong k^N$ (so $N = \dim V$), with relations

$$\alpha a_{jl} = (\alpha a)_{jl},\ a_{jl} + b_{jl} = (a+b)_{jl},\ \sum_m a_{jm} b_{ml} = (ab)_{jl},\ 1_{jl} a_{j'l'} = \delta_{lj'} a_{j'l'}, \tag{1.1.1}$$

for all $a, b \in A$ and $\alpha \in k$.

M. Kontsevich and A. Rosenberg [59] proposed the principle that the family of representation schemes $\{\mathrm{Rep}(A,V)\}$, parametrized by the finite-dimensional vector spaces V, for a fixed associative algebra A, should be thought of as a substitute (or "approximation") for a hypothetical non-commutative affine scheme "Spec(A)". According to this principle, for a property or structure on A to have a geometric meaning, it should naturally induce the corresponding geometric property or structure on $\mathrm{Rep}(A,V)$ for all V. This point of view provides a test to check the validity of the definitions to be proposed as non-commutative analogues of classical geometric notions.

In this introduction, we will work over a finite-dimensional semisimple associative algebra R over a field k of characteristic zero. In particular, A will be an associative R-algebra. A proposal for the space of 'regular functions' on A satisfying the Kontsevich–Rosenberg principle is the vector space $A/[A,A]$ (see e.g. [41, Definition 11.3.1]). More interestingly, following J. Cuntz and D. Quillen [31],

$$\Omega^\bullet_R A := \mathrm{T}_A\,\Omega^1_R A$$

is called *the algebra of non-commutative differential forms* of A (relative over R), where $\mathrm{T}_A(-)$ means tensor algebra over A, and the A-bimodule of *non-commutative differential 1-forms* $\Omega^1_R A$, endowed with certain R-linear derivation $\mathrm{d}\colon A\to\Omega^1_R A$ (called the *de Rham differential*), satisfies the following universal property: for every A-bimodule M and R-linear derivation $\theta\colon A\to M$, there exists a unique A-bimodule morphism $i_\theta\colon\Omega^1_R A\to M$ making the following diagram commute:

(1.1.2)
$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "\mathrm{d}"] & M \\ \Omega^1_R A \arrow[ur, "i_\theta"] & \end{tikzcd}$$

Since $\Omega^\bullet_R A$ does not have an interesting cohomology theory (see [41]), the non-commutative de Rham complex of A (also called the Karoubi–de Rham complex) is defined as the cochain complex $\mathrm{DR}^\bullet_R(A)=\Omega^\bullet_R A/[\Omega^\bullet_R A,\Omega^\bullet_R A]$, where $[-,-]$ denotes the super-commutator. One can use a natural evaluation map (see §2.6) on differential forms that maps the Karoubi-de Rahm complex of A to the ordinary de Rham complex of the representation schemes to conclude that the Kontsevich–Rosenberg principle holds in this case.

To address the question of which objects should be non-commutative vector fields fulfilling the Kontsevich–Rosenberg principle, one might define them simply as derivations $A\to A$. However, W. Crawley-Boevey [28] showed that when A is the coordinate ring of a smooth affine curve, the algebra of differential operators for A can be constructed using double derivations, i.e. derivations $\Theta\colon A\to A\otimes A$ (unadorned tensor products are over the base field k), rather than ordinary derivations $A\to A$. This motivates a second view point, where vector fields on A should be elements of the A-bimodule of double derivations

$$\mathbb{D}\mathrm{er}_R\, A := \mathrm{Der}_R(A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}),$$

where $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ is the enveloping algebra of A, A^{op} being the opposite algebra of A, and ${}_{A^{\mathrm{e}}}A^{\mathrm{e}}$ is A^{e} viewed as a (left) A^{e}-module by left multiplication. Then each $\Theta \in \mathbb{D}\mathrm{er}_R\, A$ induces *matrix valued vector fields* $(\Theta_{ij})_{i,j=1,\dots,N}$ on all $\mathrm{Rep}(A, V)$, so $\Theta_{ij}(a_{uv})$ depends on four indices (with a_{uv} as in (1.1.1)), and is explicitly given by

$$\Theta_{ij}(a_{uv}) = \Theta(a)'_{uj}\Theta''_{iv},$$

where by convention, we write an element x of $A \otimes A$ as $x' \otimes x''$, dropping the summation sign. Following Van den Bergh [96], this arrangement of indices will be called the *standard index convention*. Note that the universal property in (1.1.2) applied to $M = A \otimes A$ determines a canonical isomorphism of A-bimodules

$$\mathbb{D}\mathrm{er}_R\, A \xrightarrow{\cong} \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) \colon \quad \Theta \longmapsto i_\Theta. \tag{1.1.3}$$

To develop a consistent geometric theory, we would like to have a non-commutative analogue of the cotangent bundle. Following an idea of W. Crawley-Boevey [28], exploited by W. Crawley-Boevey, P. Etingof and V. Ginzburg ([30], §5), we define

$$\mathrm{T}^* A := \mathrm{T}_A\, \mathbb{D}\mathrm{er}_R\, A,$$

and view this graded algebra as the coordinate ring of the "non-commutative cotangent bundle" on the hypothetical non-commutative affine scheme "Spec(A)". It can be shown [30] that if A is smooth in an appropriate sense (used by Cuntz–Quillen [31]), the above non-commutative cotangent bundle satisfies the Kontsevich–Rosenberg principle, that is, the representation functor takes the algebra T^*A into the cotangent bundle on the representation scheme of A.

Functions, non-commutative differential forms, double derivations and the non-commutative cotangent bundle play a prominent role in this version of non-commutative algebraic geometry, but one is also interested in finding non-commutative analogues to standard geometric structures. A bi-symplectic form (in the sense of W. Crawley-Boevey, P. Etingof and V. Ginzburg [30]) is a two-form $\omega \in \mathrm{DR}^2_R(A)$ such that $\mathrm{d}\,\omega = 0$ and

$$\iota(\omega) \colon \mathbb{D}\mathrm{er}_R\, A \xrightarrow{\cong} \Omega^1_R A \colon \quad \Theta \longmapsto m \circ (i_\Theta \omega)^\circ = (i''_\Theta \omega)(i'_\Theta \omega), \tag{1.1.4}$$

is an isomorphism, where $m \colon A \otimes_R A \to A \colon\ (a, b) \mapsto ab$ is the multiplication map and $(a \otimes b)^\circ = b \otimes a$, for $a, b \in A$. In §2.6, we will explain how a bi-symplectic form $\omega \in \mathrm{DR}^2_R(A)$ induces a symplectic form on $\mathrm{Rep}(A, V)$ (see [30] for more details).

Another interesting problem is to determine what kind of structure on A induces Poisson structures on all $\mathrm{Rep}(A, V)$. Recall that a Poisson structure on a commutative algebra A is a Lie bracket $\{-,-\} : A \times A \to A$ satisfying the Leibniz rule $\{ab,c\} = a\{b,c\} + \{a,c\}b$ for all $a,b,c \in A$. For non-commutative algebras, this definition is too restrictive, because if A is a non-commutative domain (more generally, a prime ring), any Poisson bracket on A is a multiple of the commutator $[a,b] = ab - ba$ ([36], Theorem 1.2). M. Van den Bergh [96] found a less restrictive notion, which induces the usual Poisson brackets on the representation spaces. First, he defined a double bracket as an R-bilinear map $\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$ that is a double derivation in its second argument, such that $\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^\circ$ for all $a,b \in A$. If it satisfies a natural analogue of the Jacobi identity, called the *double Jacobi identity* (see (2.3.3)), then A is called a double Poisson algebra, because, if A is a smooth algebra, it satisfies the Kontsevich–Rosenberg principle:

Theorem 1.1.5 ([96], Proposition 1.2)**.** *If $(A, \{\!\{-,-\}\!\})$ is a double Poisson algebra then A_V is a Poisson algebra, with Poisson bracket given by*

$$\{a_{ij}, b_{uv}\} = \{\!\{a,b\}\!\}'_{uj}\, \{\!\{a,b\}\!\}''_{iv}\,.$$

1.2 Courant algebroids

The origins of Courant algebroids can be found in the work of T. Courant and A. Weinstein [25], who formalized certain brackets defined in physics by P. A. M. Dirac [32] in his study of constrained systems in mechanics and field theories. It was also implicit in contemporaneous work of I. Y. Dorfman [34]. Two years later, T. Courant defined in his thesis [26] a bracket on the direct sum $\mathrm{T} \oplus \mathrm{T}^*$ of the tangent and the cotangent bundles over a fixed C^∞ manifold M, given by

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}\,\mathrm{d}\,(i_X\eta - i_Y\xi), \tag{1.2.1}$$

for sections $X+\xi$ and $Y+\eta$ of $\mathrm{T} \oplus \mathrm{T}^*$. Since the Courant bracket restricts to the usual Lie bracket $[X,Y]$ on vector fields X, Y, following [45] we observe that

$$\pi([A,B]) = [\pi(A), \pi(B)], \tag{1.2.2}$$

for all sections A and B of $\mathrm{T} \oplus \mathrm{T}^*$, where $\pi\colon \mathrm{T} \oplus \mathrm{T}^* \to \mathrm{T}$ is the canonical projection. However $(\mathrm{T} \oplus \mathrm{T}^*, [-,-])$ is not a Lie algebroid, because it only satisfies the Jacobi identity up to an exact term. More precisely, defining the *Jacobiator* as a trilinear operator that measures the failure to satisfy the Jacobi identity, i.e.,

$$\mathrm{Jac}(A,B,C) = [[A,B],C] + [[B,C],A] + [[C,A],B]$$

for all sections A, B, C of $\mathrm{T} \oplus \mathrm{T}^*$, one can show that

$$\mathrm{Jac}(A,B,C) = \mathrm{d}\,(\mathrm{Nij}(A,B,C)), \tag{1.2.3}$$

where

$$\mathrm{Nij}(A,B,C) = \frac{1}{3}(\langle [A,B],C\rangle + \langle [B,C],A\rangle + \langle [C,A],B\rangle)$$

is defined using the canonical inner product on $\mathrm{T} \oplus \mathrm{T}^*$, given by

$$\langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y) + \eta(X)). \tag{1.2.4}$$

The identity (1.2.3) is proved by applying well-known identities relating the Lie derivative and the contraction operator:

$$\mathrm{L}_X = \mathrm{i}_X \,\mathrm{d} + \mathrm{d}\, \mathrm{i}_X, \quad \mathrm{L}_{[X,Y]} = [\mathrm{L}_X, \mathrm{L}_Y], \quad \mathrm{i}_{[X,Y]} = [\mathrm{L}_X, \mathrm{i}_Y], \tag{1.2.5}$$

for all vector fields X, Y. These identities also imply

$$[A, fB] = f[A,B] + (\pi(A)f)B - \langle A, B\rangle \,\mathrm{d}f \tag{1.2.6}$$

$$\pi(A)\langle B, C\rangle = \langle [A,B] + \mathrm{d}\,\langle A,B\rangle, C\rangle + \langle B, [A,C] + \mathrm{d}\,\langle A,C\rangle\rangle. \tag{1.2.7}$$

The identities (1.2.2), (1.2.3), (1.2.6), (1.2.7) make $(\mathrm{T}\oplus\mathrm{T}^*, \langle -,-\rangle, [-,-], \pi)$ into the motivating example of a Courant algebroid, as first introduced by Liu, Weinstein and Xu [70]. Formally, a *Courant algebroid* is a vector bundle F equipped with a non-degenerate symmetric bilinear form $\langle -,-\rangle$, a skew-symmetric bracket $[-,-]$ on $C^\infty(F)$, and a smooth bundle map $\pi\colon F \to \mathrm{T}$ called the anchor, satisfying certain compatibility conditions that generalize (1.2.2), (1.2.3), (1.2.6), (1.2.7). Here, $C^\infty(F)$ is the space of smooth sections of F. By adding a symmetric term, as already suggested in [70], D. Roytenberg [80] twisted the bracket of a Courant algebroid, sacrificing skew-symmetry, but obtaining an equivalent (and in a sense more natural) definition of a Courant algebroid, where the Jacobi identity for this non skew-symmetric bracket resembles a Leibniz rule. Since their introduction, Courant algebroids have attracted substantial attention in mathematics and physics, stimulated by the generalized complex geometry introduced by N. Hitchin [49] and further developed by M. Gualtieri [45], and because they may be the right framework for certain classes of string theory, as pointed out by P. Ševera [88] (see also [6]).

A Courant algebroid $(E, \pi, \langle -,-\rangle, [-,-])$ is *exact* if π induces an exact sequence

$$0 \longrightarrow \mathrm{T}^* \xrightarrow{\pi^*} E \xrightarrow{\pi} \mathrm{T} \longrightarrow 0. \tag{1.2.8}$$

On any exact Courant algebroid, one can always choose a right splitting $\nabla\colon \mathrm{T} \to E$ that is isotropic, that is, its image in E is isotropic with respect to $\langle -,-\rangle$. The curvature 3-form $H \in \Omega^3_{\mathrm{cl}}(M)$ of this splitting is defined by

$$\mathrm{i}_Y \mathrm{i}_X H = 2s[\nabla(X), \nabla(Y)],$$

where $s\colon E \to \mathrm{T}^*$ is the induced left splitting and X, Y are vector fields on M. Then the cohomology class $[H] \in \mathrm{H}^3(M, \mathbb{R})$, called the Ševera class, is independent of the splitting, as isotropic splittings of (1.2.8) differ by 2-forms $b \in \Omega^2(M)$, and a change of splitting modifies the curvature by the exact form $\mathrm{d}b$. In fact, the Ševera class determines the exact Courant algebroid structure on E, up to isomorphism.

Using P. Ševera's classification of exact Courant algebroids [88], a possible approach to define non-commutative Courant algebroids might be given, at least in the exact case, by the non-commutative analogue of the standard Courant algebroid $\mathrm{T}\oplus\mathrm{T}^*$, with the tangent bundle T replaced by the bimodule $\mathbb{D}\mathrm{er}_R\, A$ of double derivations, and the cotangent bundle T^* replaced by the bimodule $\Omega^1_R A$ of differential forms. Furthermore, one might define the Courant bracket combining the de Rham differential [31], certain non-commutative analogues of the Lie derivative and the contraction operator of a double derivation with differential forms [30], and the double Schouten–Nijenhuis [96]. However, this direct attempt is not satisfactory because, to check the Kontsevich–Rosenberg principle in the representation spaces, we need non-commutative versions of the identities (1.2.5), that so far have not been proved in this setting of non-commutative algebraic geometry.

1.3 Symplectic $\mathbb{N}Q$-manifolds

An alternative approach to define non-commutative Courant algebroids is motivated by graded geometry. An $\mathbb{N}$-*graded manifold* (or $\mathbb{N}$-*manifold*, for short) $\mathcal{M}$ of weight n and dimension $(p; r_1, ..., r_n)$ is a smooth p-dimensional manifold M endowed with a sheaf $C^\infty(\mathcal{M})$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras, that is locally isomorphic to a (graded) polynomial ring $C^\infty_U(M)[\xi^1_1, \ldots, \xi^{r_1}_1, \xi^1_2, \ldots, \xi^{r_2}_2, \ldots, \xi^1_n, \ldots, \xi^{r_n}_n]$, for open subsets $U \subset M$, where ξ^j_i are variables of weight i (where the grading is called weight). The graded structure of $C^\infty(\mathcal{M})$ determines a graded Euler vector field Eu on $\mathcal{M}$, that acts on vector fields and differential forms on $\mathcal{M}$ via the Lie derivative, whereby objects such as symplectic and Poisson structures also acquire weights. In particular, a *symplectic structure* of weight n is a closed non-degenerate 2-form ω such that $\mathrm{L}_{\mathrm{Eu}}\omega = n\omega$.

Inspired by independent unpublished observations by Y. Kosmann-Schwarzbach, P. Ševera and P. Xu on the relationship of derived brackets with the Courant brackets (see e.g. [61, §3.4]), and A. Yu. Vaintrob [94], who interpreted Lie algebroids as odd self-commuting vector fields on a supermanifold, D. Roytenberg [81], following ideas of Ševera [88, 89], proved that Courant algebroids are equivalent to symplectic $\mathbb{N}Q$-manifolds of weight 2. Here, an $\mathbb{N}Q$-*manifold* $(\mathcal{M}, Q)$ is an $\mathbb{N}$-manifold $\mathcal{M}$ endowed with an integrable homological vector field Q of weight $+1$ ("homological" means $[Q, Q] = 2Q^2 = 0$, where $[-, -]$ is the graded commutator), and a *symplectic* $\mathbb{N}Q$-*manifold* $(\mathcal{M}, \omega, Q)$ is an $\mathbb{N}Q$-manifold whose homological vector field is compatible with a symplectic form ω, that is, $\mathrm{L}_Q\omega = 0$, where L_Q is the Lie derivative along the homological vector field Q.

Theorem 1.3.1 ([81], Theorem 3.3 & Theorem 4.5)**.**

(i) *Symplectic* $\mathbb{N}$*-manifolds of weight 2 are in one-to-one correspondence with pseudo-Euclidean vector bundles.*

(ii) *Symplectic* $\mathbb{N}Q$*-manifolds of weight 2 are in 1-1 correspondence with Courant algebroids.*

Symplectic $\mathbb{N}Q$-manifolds are generalizations of PQ-manifolds on supermanifolds, introduced by A. Schwarz [85] as a geometric version of the formalism developed by I. Batalin and G. Vilkovisky [10] in physics to quantize classical field theories in the Lagrangian formalism. Here, a P-structure is an odd-symplectic structure and a Q-structure is a nilpotent vector field given by the odd-Poisson bracket with an action functional. From this view point, the $\mathbb{N}$-grading can be viewed as an enhancement of the $\mathbb{Z}/2$-grading to keep track of the ghost number.

We should also mention that D. Roytenberg [81] also classified $\mathbb{N}Q$-manifolds of weight 1. This result has applications in two-dimensional Topological Field Theory.

Theorem 1.3.2 ([81], Proposition 3.1 & Proposition 4.1)**.**

(i) *Symplectic $\mathbb{N}$-manifolds of weight 1 are in 1-1 correspondence with ordinary smooth manifolds. The correspondence attaches to each smooth manifold N, the symplectic $\mathbb{N}$-manifold* $(\mathrm{T}^*[1]N, \omega)$*, where ω is determined by the Schouten bracket of multivector fields.*

(ii) *Symplectic $\mathbb{N}Q$-manifolds of weight 1 are in 1-1 correspondence with ordinary Poisson manifolds.*

1.4 Bi-symplectic $\mathbb{N}Q$-algebras

Theorem 1.3.1 suggests a strategy to define non-commutative Courant algebroids satisfying the Kontsevich–Rosenberg principle. Namely, in this thesis we will adapt D. Roytenberg's constructions [81] — based on P. Ševera's insights [88] — to a version of non-commutative algebraic geometry where the bi-symplectic structures [30] and the double Poisson structures [96] will be the main cornerstones replacing the corresponding standard geometric structures. In this approach, our first aim is to define suitable non-commutative analogues of symplectic $\mathbb{N}Q$-manifolds.

A tensor $\mathbb{N}$-algebra is an $\mathbb{N}$-graded associative algebra that is the tensor algebra of a positively graded bimodule M, whose underlying ungraded bimodule is projective and finitely generated over the weight-zero subalgebra $A^0 \subset A$. A bi-symplectic $\mathbb{N}Q$-algebra of weight N is a tensor $\mathbb{N}$-algebra of weight N endowed with a bi-symplectic form ω of weight N and a bi-symplectic double derivation Q of weight $+1$, with $\{\!\{Q, Q\}\!\} = 0$, where $\{\!\{-, -\}\!\}$ is the canonical double Schouten–Nijenhuis bracket on double derivations. To construct these objects, we will need generalizations to graded associative algebras of tools introduced in [30, 31, 96] — these foundations are the contents of Chapter 2.

The next steps followed in this thesis to construct non-commutative Courant algebroids will be, according to D. Roytenberg's proof of Theorem 1.3.1, as follows:

(a) Start with a bi-symplectic $\mathbb{N}Q$-algebra (A, ω, Q) of weight 2.

(b) Show that the underlying bi-symplectic tensor $\mathbb{N}$-algebra A of weight 2 is determined by a pair $(E, \langle -, - \rangle)$ consisting of a projective finitely generated A^0-bimodule E, endowed with a symmetric non-degenerate pairing $\langle -, - \rangle$.

(c) Use the double derivation Q to determine a bracket $[\![-,-]\!]$ on E and an anchor $\rho\colon E \to \mathbb{D}\mathrm{er}_R(A^0)$.

(d) As a conclusion, from a bi-symplectic $\mathbb{N}Q$-algebra of weight 2, construct a non-commutative Courant algebroid, defined a 4-tuple $(E, \langle -,-\rangle, [\![-,-]\!], \rho)$.

To shorten notation, hereafter we define $B = A^0$ — an ungraded subalgebra of A.

1.5 Bi-symplectic $\mathbb{N}Q$-algebras of weight 1 and double Poisson algebras

As in the case of manifolds, the classification of bi-symplectic $\mathbb{N}Q$-algebras of weight 1 provides a preliminary test to examine the tools introduced so far (this classification is carried out in Chapter 4), and furthermore, provides new insights into the structure of M. Van den Bergh's double Poisson algebras.

Theorem 1.5.1 (Theorems 4.2.1 and 4.3.2)**.**

(i) *Bi-symplectic tensor smooth $\mathbb{N}$-algebras of weight 1 are in 1-1 correspondence, up to isomorphism, with smooth associative R-algebras. The correspondence assigns to each smooth associative R-algebra B, the pair (A,ω) consisting of the tensor $\mathbb{N}$-algebra*

$$A = \mathrm{T}^*[1]B := \mathrm{T}_B(\mathbb{D}\mathrm{er}_R\, B[-1])$$

and the bi-symplectic form ω determined by the double Schouten–Nijenhuis bracket of the tensor algebra of the B-bimodule of double derivations over B.

(ii) *Bi-symplectic $\mathbb{N}Q$-algebras of weight 1 are in 1-1 correspondence, up to isomorphism, with double Poisson algebras.*

The main technical result used in the proof is a graded non-commutative version in weight 0 of the Darboux theorem in symplectic geometry. As a similar result will be needed for weight 2, we will show a more general result, valid for bi-symplectic tensor $\mathbb{N}$-algebras (A,ω) of arbitrary weight N over a smooth associative R-algebra B. By definition, $A = \mathrm{T}_B\, M$ is a tensor algebra of a positively graded B-bimodule

$$M := M_1 \oplus \cdots \oplus M_N,$$

where $M_i = E_i[-i] \subset M$ is the homogeneous B-sub-bimodule of weight i, for $i = 1,\ldots,N$. Here, $V[-j]_i := V_{i-j}$ for a graded vector space or (bi)module V and $j \in \mathbb{Z}$, so E_i are B-bimodules of weight 0 (see §2.1). As a bi-symplectic form $\omega \in \mathrm{DR}^2_R(A)$ has weight N, it determines an A-bimodule isomorphism $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R\, A \xrightarrow{\cong} \Omega^1_R A[-N]$ (cf. (1.1.4)); in Theorem 3.2.2, we show that it restricts to a B-bimodule isomorphism

$$\widetilde{\iota}(\omega)_{(0)}\colon \mathbb{D}\mathrm{er}_R\, B \xrightarrow{\cong} E_N. \tag{1.5.2}$$

1.6 Bi-symplectic $\mathbb{N}$-algebras of weight 2

Our aim in Chapter 5 is to describe bi-symplectic tensor $\mathbb{N}$-algebras (A,ω) of weight 2 satisfying the above condition (a). On the one hand, by definition, $A := \mathrm{T}_B\, M$ with $M := E_1[-1] \oplus E_2[-2]$, for weight-zero B-bimodules E_1 and E_2, so A has B-sub-bimodules of weights $0, 1, 2$, given by

$$A^0 = B, \quad A^1 = E_1, \quad A^2 = (E_1 \otimes_B E_1) \oplus E_2,$$

and hence we have a trivial B-bimodule short exact sequence

$$0 \longrightarrow E_1 \otimes_B E_1 \longrightarrow A^2 \longrightarrow E_2 \longrightarrow 0.$$

On the other hand, the bi-symplectic form $\omega \in \mathrm{DR}^2_R(A)$ of weight 2 induces a double Poisson bracket $\{\!\{-,-\}\!\}_\omega$ of weight -2, providing a family of double derivations and a family of double differential operators defined, respectively, as

$$\begin{aligned}
&\mathbb{X}\colon A^2 \longrightarrow \mathbb{D}\mathrm{er}_R\, B\colon\ a \longmapsto (\mathbb{X}_a := \{\!\{a,-\}\!\}_\omega\,|_B\colon B \longrightarrow B)\\
&\mathbb{D}\colon A^2 \longrightarrow \mathbb{E}\mathrm{nd}_{R^{\mathrm{e}}}(E_1)\colon\ a \longmapsto (\mathbb{D}_a := \{\!\{a,-\}\!\}_\omega\,|_{E_1}\colon E_1 \longmapsto E_1 \otimes B \oplus B \otimes E_1)\,,
\end{aligned}$$

where $\mathbb{E}\mathrm{nd}_{R^{\mathrm{e}}}(E_1) := \mathrm{Hom}_{R^{\mathrm{e}}}(E_1, E_1 \otimes B \oplus B \otimes E_1)$.

Furthermore, $\{\!\{-,-\}\!\}_\omega$ restricts to a pairing $\langle -,-\rangle$ on E_1 (in the sense of M. Van den Bergh [97]), that is symmetric in the sense that $\langle e_1, e_2\rangle = \langle e_2, e_1\rangle^\circ$ for all $e_1, e_2 \in E_1$. Following D. Roytenberg's method, we need to show this pairing is non-degenerate (i.e. it induces an isomorphism between E_1 and its bidual $E_1^\vee := \mathrm{Hom}_{B^{\mathrm{e}}}(E_1, {}_{B^{\mathrm{e}}}B^{\mathrm{e}})$). This result is achieved using a Darboux-type theorem in weight 1, obtained in the framework of double graded quivers, whose basic structure is explained in §3.3.1. A *double graded quiver* $\overline{P}$ is obtained from a graded quiver P of weight $|P| = N$ by adjoining a reverse arrow $a^*\colon j \to i$ for each arrow $a\colon i \to j$ in P and whose weight is $|a^*| = N - |a|$ (see Definition 3.3.4). To a double graded quiver $\overline{P}$ (of even weight N), we can attach a bi-symplectic tensor $\mathbb{N}$-algebra A of weight N, defined simply as the graded path algebra of $\overline{P}$, with the bi-symplectic form $\omega = \sum_{a \in P_1} \mathrm{d}a\, \mathrm{d}a^*$ of weight N. In this case, $A^0 = B$ is the path algebra of the weight 0 subquiver of $\overline{P}$. In Theorem 3.3.40, we prove that when $N = 2$, the isomorphism $\iota(\omega)$ restricts, in weight 1, to an isomorphism $\flat\colon E_1 \to E_1^\vee$. This map enables us to define a symmetric non-degenerate pairing $\langle -,-\rangle$ on E_1, that coincides with the restriction $\{\!\{-,-\}\!\}_\omega\,|_{E_1 \otimes E_1}\colon E_1 \otimes E_1 \to B \otimes B$.

Using the pair $(E_1, \langle -,-\rangle)$, we can construct a non-commutative analogue $\mathbb{A}\mathrm{t}(E_1)$ of the Atiyah algebroid, called the metric double Atiyah algebra, defined as the space consisting of pairs $(\mathbb{X}, \mathbb{D})$ with $\mathbb{X} \in \mathbb{D}\mathrm{er}_R\, B, \mathbb{D} \in \mathbb{E}\mathrm{nd}_{R^{\mathrm{e}}}(E_1)$, such that

$$\mathbb{D}(be) = b\mathbb{D}(e) + \mathbb{X}(b)e, \quad \mathbb{D}(eb) = \mathbb{D}(e)b + e\mathbb{X}(b),$$

for all $b \in B$, $e \in E$, and which, in addition, preserve the pairing $\langle -,-\rangle$, that is,

$$\sigma_{(123)}\mathbb{X}(\langle e_2, e_1\rangle) = \langle e_1, \mathbb{D}(e_2)\rangle_L + \sigma_{(132)}\langle e_2, \mathbb{D}(e_1)^\circ\rangle_L,$$

for all $e_1, e_2 \in E_1$, where $\langle -,-\rangle_L$ is a canonical extension of $\langle -,-\rangle$. Then $\mathbb{A}\mathrm{t}(E_1)$, equipped with a bracket (5.5.8) and the anchor $\rho : \mathbb{A}\mathrm{t}(E_1) \to \mathbb{D}\mathrm{er}_R\, B \colon (\mathbb{X}, \mathbb{D}) \mapsto \mathbb{X}$, is a double Lie–Rinehart algebra, that is, a non-commutative analogue of a Lie–Rinehart algebra, which itself is the algebraic analogue of a Lie algebroid. Furthermore, there is a B-bimodule short exact sequence

$$0 \longrightarrow \mathbb{a}\mathrm{d}_{B^{\mathrm{e}}}(E_1) \longrightarrow \mathbb{A}\mathrm{t}(E_1) \xrightarrow{\rho} \mathbb{D}\mathrm{er}_R\, B \longrightarrow 0,$$

where $\mathbb{a}\mathrm{d}_{B^{\mathrm{e}}}(E_1)$ is the space of $\mathbb{D} \in \mathbb{E}\mathrm{nd}_{R^{\mathrm{e}}}(E_1)$ with $(0, \mathbb{D}) \in \mathbb{D}\mathrm{er}_R\, B$ (cf. (5.7.3)).

Following D. Roytenberg, we can now try to show that the double Lie–Rinehart algebra $\mathbb{A}\mathrm{t}(E_1)$ is isomorphic to A^2, with the bracket obtained by restriction of the Poisson bracket. However, $\{\!\{a, a'\}\!\}_\omega \in (A \otimes A)^2 = E_2 \otimes B \oplus B \otimes E_2 \oplus E_1 \otimes E_1$, for all $a, a' \in A^2$, so to construct this isomorphism, it is useful to consider a larger class of 'twisted' double Lie–Rinehart algebras. These are B-bimodules N, equipped with a pair $(\overline{N}, \langle -,-\rangle_{\overline{N}})$ consisting of a B-sub-bimodule $\overline{N}$ and a non-degenerate symmetric pairing on $\overline{N}$, and an R-bilinear 'twisted double bracket' $\{\!\{-,-\}\!\}_N$ on N, such that $\{\!\{n_1, n_2\}\!\}_N \in N \otimes B \oplus B \otimes N \oplus \overline{N} \otimes \overline{N}$, for all $n_1, n_2 \in N$, satisfying suitable axioms. With this definition, A^2 is a twisted double Lie–Rinehart algebra, and the families $\mathbb{X}$ and $\mathbb{D}$ determine a map of twisted double Lie–Rinehart algebras

$$\Psi \colon A^2 \longrightarrow \mathbb{A}\mathrm{t}(E_1) \colon \quad a \longmapsto (\mathbb{X}_a, \mathbb{D}_a). \tag{1.6.1}$$

Furthermore, Ψ determines a commutative diagram (see (5.7.2))

$$\begin{tikzcd}
0 \arrow[r] & E_1 \otimes_B E_1 \arrow[r] \arrow[d, "\Psi|_{E_1 \otimes_B E_1}"] & A^2 \arrow[r] \arrow[d, "\Psi"] & E_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{a}\mathrm{d}_{B^{\mathrm{e}}}(E_1) \arrow[r] & \mathbb{A}\mathrm{t}(E_1) \arrow[r, "\rho"] & \mathbb{D}\mathrm{er}_R\, B \arrow[r] \arrow[u, "\widetilde{\iota}(\omega)_{(0)}"'] & 0
\end{tikzcd}$$

where the rows are short exact sequences of B-modules. Furthermore, the right-hand vertical map, induced by $\iota(\omega)$, is an isomorphism

$$\widetilde{\iota}(\omega)_{(0)} \colon \mathbb{D}\mathrm{er}_R\, B \xrightarrow{\cong} E_2,$$

by Theorem 3.2.2 (see (1.5.2)). Regarding the left-hand vertical arrow of this commutative diagram, obtained by restriction of Ψ to $E_1 \otimes_B E_1$, we describe in §5.7 explicit basis of the B-bimodules $E_1 \otimes_B E_1$ and $\mathbb{a}\mathrm{d}_{B^{\mathrm{e}}}(E_1)$, using the structure of the double graded quiver $\overline{P}$, and show that this arrow maps each basis bijectively onto each other, and so it is also an isomorphism. Hence we conclude that the map Ψ in (1.6.1) is an isomorphism. This implies the main result of Chapter 5.

Theorem 1.6.2 (Theorem 5.7.1)**.** *Let (A, ω) be a pair consisting of the graded path algebra of a double quiver $\overline{P}$ of weight 2, and the bi-symplectic form $\omega \in \mathrm{DR}^2_R(A)$ of weight 2 defined in §3.3.4. Let B be the path algebra of the weight 0 subquiver of $\overline{P}$. Then (A, ω) is completely determined by the pair $(E_1, \langle -,-\rangle)$ consisting of the*

B-bimodule E_1, with basis consisting of weight 1 paths in $\overline{P}$, and the symmetric non-degenerate pairing

$$\langle -,-\rangle := \{\!\!\{-,-\}\!\!\}_\omega\,|_{E_1\otimes E_1} \longrightarrow B\otimes B.$$

1.7 Bi-symplectic $\mathbb{N}Q$-algebras of weight 2 and double Courant–Dorfman algebras

In Chapter 6, we focus on the construction of non-commutative Courant algebroids, using the pairs $(E_1,\langle -,-\rangle)$ of Theorem 1.6.2. More precisely, a double pre-Courant–Dorfman algebra over the R-algebra B is a 4-tuple $(E,\langle -,-\rangle,\rho,[\![-,-]\!])$ consisting of a projective finitely generated B-bimodule E endowed with a symmetric non-degenerate pairing (the *inner product*)

$$\langle -,-\rangle\colon E\otimes E\longrightarrow B\otimes B,$$

a B-bimodule morphism

$$\rho\colon E\longrightarrow \mathbb{D}\mathrm{er}_R\, B, \tag{1.7.1}$$

called the *anchor*, and an operation

$$[\![-,-]\!]\colon E\otimes E\longrightarrow (E\otimes B)\oplus(B\otimes E), \tag{1.7.2}$$

called the *double Dorfman bracket*, which is R-linear for the left B^{e}-module structure on B^{e} in the second argument and R-linear for the right B^{e}-module structure on B^{e} in the first argument. These data must satisfy certain compatibility conditions (see (6.3.4) in Definition 6.3.1). In addition, if the double pre-Courant–Dorfman algebra satisfies the "double Jacobi–Courant rule" (6.3.5), the 4-tuple $(E,\langle -,-\rangle,\rho,[\![-,-]\!])$ is called a double Courant–Dorfman algebra.

As in the commutative case, given a bi-symplectic $\mathbb{N}Q$-algebra (A,ω,Q) of weight 2, the homological double derivation can be written as $Q=\{\!\!\{S,-\}\!\!\}_\omega$, where $S\in A^3$ enables us to recover the structure of double pre-Courant–Dorfman algebra using derived brackets in this framework (see Proposition 6.4.2) by the formulae

$$\begin{aligned}\rho(e_1)(b) &:= \{\!\!\{\{S,e_1\}_\omega,b\}\!\!\}_\omega\,,\\ [\![e_1,e_2]\!] &:= \{\!\!\{S,e_1\}_\omega,e_2\}\!\!\}_\omega\,,\end{aligned}$$

for all $b\in B$ and $e_1,e_2\in E_1$, where $\{-,-\}_\omega = m\circ\{\!\!\{-,-\}\!\!\}_\omega$ is the *associated bracket* in A (see (2.3.5)). Then the condition $\{\!\!\{Q,Q\}\!\!\}=0$ implies $\{S,S\}_\omega=0$, whereas the latter condition implies the "double Jacobi–Courant identity" in (6.3.5), by Proposition 6.4.6, so we obtain a double Courant–Dorfman algebra $(E_1,\langle -,-\rangle,\rho,[\![-,-]\!])$.

In conclusion, we obtain the main result of this thesis:

Theorem 1.7.3 (Theorem 6.4.8)**.** *Let (A,ω,Q) be a bi-symplectic $\mathbb{N}Q$-algebra of weight 2, where A is the graded path algebra of a double quiver $\overline{P}$ of weight 2 endowed with a bi-symplectic form $\omega\in \mathrm{DR}^2_R(A)$ of weight 2 defined in §3.3.4 and a homological double derivation Q. Let B be the path algebra of the weight 0 subquiver of $\overline{P}$, and $(E_1,\langle -,-\rangle)$ the pair consisting of the B-bimodule E_1 with basis consisting of the weight 1 paths in P and the symmetric non-degenerate pairing $\langle -,-\rangle := \{\!\{-,-\}\!\}_\omega|_{E_1\otimes E_1}\to B\otimes B$.*

Then the triple (A,ω,Q) determines an element $S\in A^3$ such that

(i) *S induces a double pre-Courant–Dorfman algebra structure on $(E_1,\langle -,-\rangle)$ by*
$$\rho(e_1)(b):=\{\!\{\{S,e_1\}_\omega,b\}\!\}_\omega\,,\quad [\![e_1,e_2]\!]:=\{\!\{\{S,e_1\}_\omega,e_2\}\!\}_\omega\,,$$
for all $b\in B, e_1,e_2\in E_1$, where $\{-,-\}_\omega = m\circ\{\!\{-,-\}\!\}_\omega$ is the associated bracket *in A.*

(ii) *The bi-symplectic $\mathbb{N}Q$-algebra (A,ω,Q) of weight 2 induces a double Courant–Dorfman algebra $(E_1,\langle -,-\rangle,\rho,[\![-,-]\!])$ over B.*

1.8 Contents of the thesis

In Chapter 2, we introduce graded versions of basics notions defined in non-commutative symplectic geometry by Crawley-Boevey, Etingof and Ginzburg [30], and in non-commutative Poisson geometry by Van den Bergh [96]. To fix notation, in §2.1 we start reviewing graded versions of well-known constructions for basic objects, such as algebras and modules, and above all, we introduce the outer and inner bimodule structures on $A\otimes A$, for a graded algebra A (see (2.1.21)). In §2.1.2 we present some classical results concerning isomorphisms of (projective finitely generated) graded A-modules. Adapting constructions of Cuntz and Quillen [31] and Crawley-Boevey, Etingof and Ginzburg [30] to a graded algebra A over an associative algebra R, in §2.2 we define the bimodules $\Omega^1_R A$ and $\mathbb{D}\mathrm{er}_R A$ of non-commutative relative differentials forms and double derivations §2.2.4, and the notion of smoothness for A over R (see Definition 2.2.22). In this thesis, the key example of a smooth algebra will be the tensor algebra satisfying suitable conditions specified in Proposition 2.2.23.

After the preliminary definitions and facts included in §2.1, §2.1.2, in §2.3–§2.5 we introduce graded versions of basic concepts in non-commutative Poisson and symplectic structures. First, in §2.3.1, we review Van den Bergh's double Poisson algebras [96], and define double Poisson graded algebras (see Definition 2.3.9). Then we define the graded double Schouten–Nijenhuis bracket (see (2.3.14)), review bi-symplectic forms and Hamiltonian double derivations (see Definition 2.4.1), introduce bi-symplectic associative graded algebras (Definition 2.5.3) and prove some basic results about them (Lemma 2.5.5 and 2.5.6). Finally, in §2.6, we review how the Kontsevich–Rosenberg principle works for bi-symplectic forms.

In Chapter 3 we shall prove two technical results of graded bi-symplectic forms, roughly speaking corresponding to graded non-commutative versions, in weights 1 and 2, of the Darboux Theorem in symplectic geometry (as explained, for instance, in [18], §8.1). They turn out to be essential in subsequent chapters. In §3.1, following [31], we introduce the cotangent exact sequence relating absolute and relative differential forms, and also study its bidual (see Lemma 3.1.10). In §3.2, we introduce the crucial notion of bi-symplectic tensor $\mathbb{N}$-algebra of weight N (here $N \in \mathbb{N}^*$) which, in particular establishes an isomorphism between the space of double derivations and the bimodule of non-commutative differential 1-forms on the tensor $\mathbb{N}$-algebra given by ω, the bi-symplectic form of weight N. The first technical result is Theorem 3.2.2. It states that if (A, ω), with $A = \mathrm{T}_B$ is a bi-symplectic tensor $\mathbb{N}$-algebra of weight N where R is a semisimple finite dimensional k-algebra, B is a smooth R-algebra and $M := E_1[-1] \oplus \cdots \oplus E_N[-N]$ for finitely generated projective B-bimodules E_i for all $1 \leq i \leq N$, the isomorphism $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R A \longrightarrow \Omega^1_R A[-N]$ restricts, in weight 0, to the B-bimodule isomorphism $\widetilde{\iota}(\omega)_{(0)}\colon \mathbb{D}\mathrm{er}_R B \xrightarrow{\cong} E_N$.

The second technical result is Theorem 3.3.40 where we carry out the construction of the isomorphism $\flat\colon E_1 \to E_1^\vee$ which turns out to be the restriction of the isomorphism $\iota(\omega)$ in weight 1. This Theorem is proved in the setting of graded double quivers (of weight 2) whose basics are reviewed in §3.3.1 (see Definition 3.3.4). In particular, since the graded path algebra of these objects can be expressed in terms of the graded tensor algebra of the bimodule V_P and as the graded tensor of the bimodule M_P (see (3.3.5) and (3.3.25)), in Lemma 3.3.7 we prove that there exists a canonical isomorphism between both descriptions. Finally, in Proposition 3.3.34, we show that graded double quivers are endowed with a canonical bi-symplectic form of even weight.

D. Roytenberg [81] proved that symplectic $\mathbb{N}Q$-manifolds of weight 1 are in 1-1 correspondence with ordinary Poisson manifolds. In Chapter 3, we extend this result to the noncommutative setting using techniques of noncommutative algebraic geometry.

Once we review Roytenberg's result in §4.1, we carry out the classification of bi-symplectic tensor $\mathbb{N}$-algebras of weight 1 (see §4.2), which are in 1-1 correspondence with smooth associative algebras. In the last section of the chapter, we introduce the essential notion of *bi-symplectic* $\mathbb{N}Q$*-algebras* (which can be regarded as the noncommutative analogues of symplectic $\mathbb{N}Q$-manifolds) and in Theorem 4.3.2 we classify them in weight 1: bi-symplectic $\mathbb{N}Q$-algebras of weight 1 are in 1-1 correspondence with double Poisson algebras.

Chapter 5 is somehow the core of this thesis. In §5.1 we sketch a result of

D. Roytenberg that can be reformulated more algebraically using Lie–Rinehart algebras (the algebraic structure corresponding to Lie algebroids) as follows: the structure of a symplectic polynomial $\mathbb{N}$-algebra of weight 2 is completely determined by a finitely generated projective B-module E_1 endowed with a symmetric non-degenerate bilinear form $\langle -, - \rangle$ (see [81], Theorem 3.3). In §5.2, if B is a smooth associative algebra, given a bi-symplectic tensor $\mathbb{N}$-algebra of weight 2 over B $A = \mathrm{T}_B(E_1[-1] \oplus E_2[-2]$ where E_1 and E_2 are projective finitely generated B-bimodules, we calculate A^0, A^1 and A^2, the subspaces $A^w \subset A$ of weights 0,1,2, and determine the structure of the double Poisson bracket of weight -2 induced by the bi-symplectic form. §5.4 is devoted to define a non-commutative counterpart of a Lie–Rinehart algebra (see Definition 5.4.1) and to prove in Proposition 5.4.9 that A^2 has this structure.

A key point in our discussion is that E_1 is endowed with a pairing (in the sense of [97], §3.1), whose definition is reviewed in (5.3.3). Using the results of §3.3, we construct a non-degenerate symmetric pairing for double graded quivers (see Lemma 5.3.7). In §5.3.4, we prove that this pairing is compatible, in a suitable sense (see §5.3.4), with certain family of "double covariant differential operators" $\mathbb{D}_a$ introduced in §5.3.2.

In §5.5 we introduce the notion of *double Atiyah algebra* and *metric double Atiyah algebra* $\mathbb{A}\mathrm{t}(E_1)$ which are endowed with brackets (5.5.8), resembling Van den Bergh's double Schouten–Nijenhuis bracket. In Proposition 5.5.10 we prove that $\mathbb{A}\mathrm{t}(E_1)$ is a double Lie–Rinehart algebra. Finally, §5.6 is devoted to prove that a map $\Psi \colon A^2 \to \mathbb{A}\mathrm{t}(E_1)$, defined in (5.6.2) using the "double covariant differential operators", is a map of twisted double Lie–Rinehart algebras (see Proposition 5.6.1). Furthermore, in §5.7 we demonstrate that, in the setting of double graded quivers, Ψ is an isomorphism and, consequently, we conclude that our bi-symplectic tensor $\mathbb{N}$-algebra A over B of weight 2 is completely determined by E_1 together with its non-degenerate symmetric pairing.

In Chapter 6, we calculate the non-commutative structures that arise when we equip a graded bi-symplectic tensor algebra (A, ω) of weight 2 with a homological double derivation Q. Here, a double derivation Q on A is homological if it satisfies the "double Maurer–Cartan" equation $\{\!\{Q, Q\}\!\} = 0$, where $\{\!\{-, -\}\!\}$ is the double Schouten–Nijenhuis commutator. Since our calculations will be based on results of Chapter 4, we focus on the case where (A, ω) is a bi-symplectic graded path algebra of a double graded quiver (see (3.3.4)). The new algebraic structures will be called "double Courant–Dorfman algebras". They are non-commutative versions of the Courant–Dorfman algebras introduced by Roytenberg [83], that themselves are to Courant algebroids what Lie–Rinehart algebras are to Lie algebroids.

In §6.1, we start with a short review of the role of Courant algebroids in geometry and physics (§6.1.1) and their definition §6.1.2. In §6.2 we provide an algebraic reformulation of Roytenberg's correspondence between symplectic $\mathbb{N}Q$-manifolds of weight 2 and Courant algebroids. Finally, in §6.3.1 we define the central object of this chapter –double Courant–Dorfman algebras–, and show that a bi-symplectic $\mathbb{N}Q$-algebra (A, ω) attached to a double graded quiver $\overline{P}$ determines a double Courant–Dorfman algebra over the path algebra of the weight 0 subquiver Q of P.

Finally, in Chapter 7, we present some questions and open directions which this thesis gives rise.

Chapter 2

Basics on (graded) non-commutative algebraic geometry

In this chapter, we introduce graded versions of basics notions defined in non-commutative symplectic geometry by Crawley-Boevey, Etingof and Ginzburg [30], and in non-commutative Poisson geometry by Van den Bergh [96]. To fix notation, in §2.1 we start reviewing graded versions of well-known constructions for basic objects, such as algebras and modules, and above all, we introduce the outer and inner bimodule structures on $A \otimes A$, for a graded algebra A (see (2.1.21)). In §2.1.2 we present some classical results concerning isomorphisms of (projective finitely generated) graded A-modules. Adapting constructions of Cuntz and Quillen [31] and Crawley-Boevey, Etingof and Ginzburg [30] to a graded algebra A over an associative algebra R, in §2.2 we define the bimodules $\Omega^1_R A$ and $\mathbb{D}\mathrm{er}_R A$ of non-commutative relative differentials forms and double derivations §2.2.4, and the notion of smoothness for A over R (see Definition 2.2.22). In this thesis, the key example of a smooth algebra will be the tensor algebra satisfying appropriate conditions specified in Proposition 2.2.23.

After the preliminary definitions and facts included in §2.1, in §2.2–§2.5 we introduce graded versions of basic concepts in non-commutative Poisson and symplectic structures. First, in §2.3.1, we review Van den Bergh's double Poisson algebras [96], and define double Poisson graded algebras (see Definition 2.3.9). Then we define the graded double Schouten–Nijenhuis bracket (see (2.3.14)), review bi-symplectic forms and Hamiltonian double derivations (see Definition 2.4.1), introduce bi-symplectic associative graded algebras (Definition 2.5.3) and prove some basic results about them (Lemma 2.5.5 and 2.5.6). Finally, in §2.6, we illustrate how the Kontsevich–Rosenberg principle works in some cases, including bi-symplectic forms.

2.1 Background on graded algebras and graded modules

Notation and conventions

We will work over a fixed (commutative) base field k. From now on, all unadorned tensor products are over the base field k. We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N} = \{0, 1, 2, ...\}$ the set of natural numbers, which by convention are the non-negative integers and by $\mathbb{N}^* = \{1, 2, 3, ...\} \subset \mathbb{N}$ the set of positive integers. Finally, if V and W are k-vector spaces, then an element $h \in V \otimes W$ is written as $h' \otimes h''$. This is a shorthand for $\sum_i h'_i \otimes h''_i$. From now on, we will use Sweedler's notation in a systematic way which consists of dropping the summation sign.

Graded vector spaces

Following [20] by a $\mathbb{Z}$-*graded vector space* (or simply, a *graded vector space*) we mean a direct sum $V = \bigoplus_{i\in\mathbb{Z}} V_i$ of vector spaces over the field k. The V_i are called the components of V of degree i. An element $v \in V$ is called *homogeneous* if $v \in V_i$ for some i, and *homogeneous of degree* i if $v \in V_i$. Finally, the degree of a homogeneous element $v \in V$ to be denoted by $|v|$. Moreover, we also denote by $V[n]$ the graded vector space with degree shifted by n, namely, $V[n] = \bigoplus_{i\in\mathbb{Z}}(V[n])_i$ with $V[n]_i = V_{i+n}$. If $f\colon V \to W$ is a map of graded vector spaces with homogeneous components $f_l\colon V_i \to W_j$ then let $f[d]\colon V[d] \to W[d]$ be the map of graded vector spaces with homogeneous components $f[d]_l\colon V[d]_i \to W[d]_j$ defined by $f[d]_l(v) = (-1)^d f_{l+d}(v)$ for $v \in V[d]_i = V_{i+d}$.

Graded rings

We say that a ring R is $\mathbb{Z}$-*graded* if there exists a family of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R such that $R = \bigoplus_{n\in\mathbb{Z}} R_n$ as abelian groups, and $R_n \cdot R_m \subset R_{n+m}$ for all homogeneous $n, m \in \mathbb{Z}$. Observe that if $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a graded ring, then R_0 is a subring of R, $1 \in R_0$ and R_n is an R_0-module for all n. From now on, all rings will be associative with 1. Let S be a graded ring. A map $f\colon R \to S$ is called a *graded ring homomorphism* if f is a ring homomorphism, $f(1_R) = 1_S$ and f respects the grading, that is, $f(R_n) \subset S_n$ for each n.

Graded (associative) algebras

An *associative graded* k-*algebra* A (for short, a *graded* k-*algebra*, *graded algebra over* k or, simply, a *graded algebra*) is a a graded ring A together with a morphism $k \to A$ (called the structure map) into its *graded centre*, $Z(A)$ whose definition is (see [65], p. 84) $\{z \in A \mid za = (-1)^{|a||z|}az$ for all homogeneous $a \in A\}$. A *morphism of graded algebras* is a morphism of graded rings that forms a commutative triangle with the structure maps over k. In a parallel way, if R is an associative unital k-algebra, we may develop the theory in the case of *graded* R-*algebras*, that is, graded algebras endowed with an algebra homomorphism $B \to A$ compatible with the identity map $R \to R$ (in particular, it is unit preserving).

Tensor product of graded algebras

In this subsection, we fix two graded algebras: A and B. Their *tensor product* is the graded algebra with underlying graded vector space $A \otimes B$, and multiplication given by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'. \tag{2.1.1}$$

The *graded opposite algebra* A^{op} is the graded algebra with the same underlying graded vector space as A and product given by

$$a^{\mathrm{op}} b^{\mathrm{op}} := (-1)^{|a||b|} (ba)^{op}, \tag{2.1.2}$$

for homogeneous $a, b \in A$, where the symbol $(-)^{\mathrm{op}}$ is used to distinguish the multiplication rules in A and A^{op}. Then there exist natural isomorphisms

$$(A^{\mathrm{op}})^{\mathrm{op}} \simeq A, \quad (A \otimes B)^{\mathrm{op}} \simeq A^{\mathrm{op}} \otimes B^{\mathrm{op}}. \tag{2.1.3}$$

The *graded enveloping algebra* of A is the graded algebra

$$A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}. \tag{2.1.4}$$

Hence the multiplication in A^{e} is given by

$$\begin{aligned}(a_1 \otimes b_1^{\mathrm{op}})(a_2 \otimes b_2^{\mathrm{op}}) &= (-1)^{|b_1||a_2|} (a_1 a_2) \otimes (b_1^{\mathrm{op}} b_2^{\mathrm{op}}) \\ &= (-1)^{|b_1||a_2| + |b_1||b_2|} (a_1 a_2) \otimes (b_2 b_1)^{\mathrm{op}},\end{aligned}$$

for homogeneous $a_i \in A$ and $b_i^{\mathrm{op}} \in A^{\mathrm{op}}$ with $i = 1, 2$. Then there exists a natural isomorphism of graded algebras

$$\begin{aligned}\tau \colon A^{\mathrm{e}} &\longrightarrow (A^{\mathrm{e}})^{\mathrm{op}} \\ a_1 \otimes a_2^{\mathrm{op}} &\longmapsto (-1)^{|a_1||a_2|} (a_2 \otimes a_1^{\mathrm{op}})^{\mathrm{op}},\end{aligned} \tag{2.1.5}$$

with inverse

$$\begin{aligned}\tau^{-1} \colon (A^{\mathrm{e}})^{\mathrm{op}} &\longrightarrow A^{\mathrm{e}} \\ (a_2 \otimes a_1^{\mathrm{op}})^{\mathrm{op}} &\longmapsto (-1)^{|a_1||a_2|} a_1 \otimes a_2^{\mathrm{op}}.\end{aligned} \tag{2.1.6}$$

It is not difficult to check that, with this definition, τ is a morphism of graded algebras.

Modules over graded algebras

Throughout, a *graded A-module* will be a *graded left A-module M*, that is, a graded vector space M endowed with a multiplication map $\mu_M : A\otimes M \to M : a\otimes m \mapsto am$, of degree zero, such that the diagrams

$$\begin{array}{ccc} A\otimes A\otimes M & \xrightarrow{\mu_A\otimes 1} & A\otimes M \\ \downarrow{\scriptstyle 1\otimes\mu_M} & & \downarrow{\scriptstyle \mu} \\ A\otimes M & \xrightarrow{\mu} & M \end{array} \qquad \begin{array}{ccc} k\otimes M & \longrightarrow & M \\ \downarrow{\scriptstyle I_A\otimes 1_A} & & \downarrow \\ A\otimes M & \xrightarrow{\mu_M} & M \end{array}$$

commute.

Given two graded A-modules M and N, an *A-module homomorphism $f : N \to M$ of degree d* is a homomorphism of k-modules, $f : N_i \to M_j$ such that $j = i + d$ and $f(\lambda n) = (-1)^{|f||\lambda|}\lambda f(n)$ for all $\lambda \in k$, $n \in N$. The set of all such f is a k-module which we denote by $\mathrm{Hom}^d_A(N, M)$ and consider

$$\mathrm{Hom}^\bullet_A(N, M) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}^d_A(N, M) = \bigoplus_{d\in\mathbb{Z}} \bigoplus_{j=i+d} \mathrm{hom}(N_i, M_j),$$

where $\mathrm{hom}(-, -)$ is the functor between (ungraded) A-modules. To shorten notation, we shall make the identification $\mathrm{Hom}_A(-, -) := \mathrm{Hom}^\bullet_A(-, -)$ for a graded algebra A; this should no cause confusion by the context. In general, $\mathrm{Hom}_A(N, M)$ does not have a graded A-module structure. However, $\mathrm{Hom}_A(M, N)$ has a structure of $Z(A)$-module by defining

$$(z \cdot f)(m) := z(f(m)).$$

A *graded right A-module M* is defined as a graded A^{op}-module. A *graded (A, B)-bimodule* is a graded $(A\otimes B^{\mathrm{op}})$-module and a *graded A-bimodule* is a graded (A, A)-bimodule, i.e. a graded A^{e}-module. The multiplication map of a right graded A-module M will often be denoted $M \otimes A \to M : m \otimes a \mapsto ma$ with $ma := (-1)^{|a||m|}a^{\mathrm{op}}m$. This definition insures that $a_1^{\mathrm{op}}(a_2^{\mathrm{op}}m) = (a_1^{\mathrm{op}}a_2^{\mathrm{op}})m$.

Graded bimodules of various types can be described in terms of various compatible graded module structures. For instance, a graded (A, B)-bimodule structure on M is equivalent to a graded left A-module structure and a graded right B-module structure on M, such that the operators of A and B commute, i.e. $(am)b = a(mb)$, where the (graded) A-module multiplication $A\otimes M \to M : a\otimes m \mapsto am$ and the (graded) B-module multiplication $M \otimes B \to M : m \otimes b \mapsto mb$ are respectively given by

(2.1.7a) $$am := (a \otimes 1_B^{\mathrm{op}})m,$$

(2.1.7b) $$mb := (-1)^{|m||b|}(1_A \otimes b^{\mathrm{op}})m.$$

The fact that these module commute means that $a(mb) = (am)b$, that is, $(-1)^{|m||b|}(a \otimes 1_B^{\mathrm{op}})(1_A \otimes b^{\mathrm{op}})m = (-1)^{(|a||b|+|m||b|)}(1_A \otimes b^{\mathrm{op}})(a \otimes 1_B^{\mathrm{op}})m.$

Hence, in particular, for $B = A$, a graded A^{e}-module M can be described by a pair of commuting graded left and right A-module structures, respectively given by

$$am := (a \otimes 1_A^{\mathrm{op}})m, \tag{2.1.8a}$$

$$ma := (-1)^{|m||b|}(1_A \otimes a^{\mathrm{op}})m. \tag{2.1.8b}$$

Similarly, a graded $(A \otimes B)$-module structure on M is equivalent to a graded A-module structure and a graded B-module structure on M, such that the operators of A and B commute, that is, $a(bm) = b(am)$, with multiplication maps given by $am := (a \otimes 1_B)m$ and $bm := (-1)^{|b||m|}(1_A \otimes b)m$.

We shall change between the above equivalent descriptions of graded left/right modules and bimodules when it is convenient; this should no cause confusion, as $(A^{\mathrm{op}})^{\mathrm{op}} = A$. Symbols such as ${}_AM$, M_A, ${}_{A,B}M$, ${}_AM_B$ or $M_{A,B}$ will indicate that M is a graded (left) A-module, a graded right A-module, a graded $(A\otimes B)$-module, a graded (A, B)-bimodule, or a graded right $(A \otimes B)$-module, respectively.

Graded modules as representations

It is sometimes convenient to identify the category of graded A-modules with the category of representations of the graded algebra A. Consider the *space of graded endomorphisms*

$$\operatorname{End}_\bullet M = \bigoplus_{l\in\mathbb{Z}} \operatorname{End}_l M = \bigoplus_{l\in\mathbb{Z}} \bigoplus_{j=i+l} \operatorname{Hom}_A(M_i, M_j),$$

on a graded vector space M, with $\operatorname{Hom}_A(-,-)$ denoting the A-module of (ungraded) homomorphisms. The multiplication of two such homogeneous maps $f_1, f_2 \colon M_\bullet \to M_\bullet$ is provided by the composition (from right to left and with the obvious compatibility between the involved degrees) $f_2 \circ f_1 \colon M \to M$ to be a homogeneous element of degree $|f_1|+|f_2|$. Again, to shorten notation, $\operatorname{End}(-) := \operatorname{End}_\bullet(-)$

A *graded representation of* A is a (unital) morphism of graded algebras $\rho\colon A \to \operatorname{End} M$ into the graded algebra. A *morphism between graded representations*, namely $f\colon \rho_1 \to \rho_2$ is a graded linear map $f\colon M_\bullet \to M'_\bullet$ such that for every i and j the diagram

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "\rho_1(a)"] & M'_i \arrow[d, "\rho_2(a)"] \\ M_j \arrow[r, "f_j"] & M'_j \end{tikzcd} \tag{2.1.9}$$

commutes for all homogeneous $a \in A$. We will often identify graded A-modules and graded A-representations using the isomorphism between their categories that to each graded A-module M assigns the representation

$$\rho_M \colon A \longrightarrow \operatorname{End} M$$

given by $\rho_M(a)m := am$, and to each representation $\rho\colon A \to \operatorname{End} M$ assigns the graded A-module M with multiplication given by $am := \rho(a)m$.

Right A^{e}-modules as A-bimodules

The category of graded right A^{e}-modules is isomorphic to the category of graded A-bimodules. To show this, it is more convenient to construct the corresponding isomorphism between the category of graded representations of $(A^{\mathrm{e}})^{\mathrm{op}}$ and A^{e}. The isomorphism assigns to each graded $(A^{\mathrm{e}})^{\mathrm{op}}$-representation $\rho\colon (A^{\mathrm{e}})^{\mathrm{op}} \to \operatorname{End} M$, its composite

$$\tau^*\rho := \rho \circ \tau\colon A^e \longrightarrow \operatorname{End} M,$$

with the graded algebra morphism τ in (2.1.5), and to each morphism $f\colon \rho_1 \to \rho_2$ of graded $(A^{\mathrm{e}})^{\mathrm{op}}$-representations $\rho_i\colon (A^{\mathrm{e}})^{\mathrm{op}} \to \operatorname{End} M_i$, the same (graded) morphism f, regarded as a morphism $\tau^*f\colon \tau^*\rho_1 \to \tau^*\rho_2$ of graded $(A^{\mathrm{e}})^{\mathrm{op}}$-representations (the fact that f is a morphism of graded representations of A^{e} follows because if the diagram (2.1.9) commutes with a replaced by u, for all homogeneous $u \in (A^{\mathrm{e}})^{\mathrm{op}}$, then this diagram also commutes with a replaced by $\tau(u)$, for all homogeneous $u \in A^{\mathrm{e}}$).

In the language of graded modules, this isomorphism assigns to each graded right A^{e}-module M, the graded A-bimodule τ^*M with underlying graded vector space M and multiplication

$$A^{\mathrm{e}} \otimes \tau^*M \longrightarrow \tau^*M\colon \quad (a_1 \otimes a_2^{\mathrm{op}}) \otimes m \longmapsto (a_1 \otimes a_2^{\mathrm{op}}) * m \tag{2.1.10}$$

given by

$$(a_1 \otimes a_2^{\mathrm{op}}) * m := \tau(a_1 \otimes a_2^{\mathrm{op}})m = (-1)^{|a_1||a_2|}(a_2 \otimes a_1^{\mathrm{op}})^{\mathrm{op}}m. \tag{2.1.11}$$

As in (2.1.8), this latter graded A-bimodules structure can be described in terms of a pair of commuting graded left and right A-module structures, respectively given by

$$a_1 * m := (a_1 \otimes 1_A^{\mathrm{op}}) * m = (1_A \otimes a_1^{\mathrm{op}})^{\mathrm{op}}m, \tag{2.1.12a}$$

$$m * a_2 := (-1)^{|a_2||m|}(1_A \otimes a_2^{\mathrm{op}}) * m = (-1)^{|a_2||m|}(a_2 \otimes 1_A^{\mathrm{op}})^{\mathrm{op}}m. \tag{2.1.12b}$$

We will often identify a graded right A^{e}-module M with the corresponding A^{e}-module τ^*M, dropping the symbol τ^*, and distinguish them with the lower indices $M_{A^{\mathrm{e}}}$ and ${}_{A^{\mathrm{e}}}M$, respectively. Finally, observe that for any graded right A^{e}-module M, its underlying ungraded right A^{e}-module is finitely generated and projective if and only if the underlying ungraded corresponding to τ^*M so is.

Transference of operators

When two graded modules have some extra operators, it is possible to transfer them to their graded tensor product or to the space of graded homomorphisms (when these spaces are defined). As usual, let A and B be graded algebras. Then, recall that $\mathrm{Hom}_A(M,N)$ is defined when M and N be both graded (left) A-modules while the graded tensor product $M \otimes_A N$ is defined when M is a right graded A-module and N is a left graded A-module.

For instance, for a graded (A,B)-bimodule M and a graded A-module N, the space $\mathrm{Hom}_A(M,N)$ of graded A-module homomorphisms is a graded (left) B-module, with multiplication $B \otimes \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,N)$ given by

$$(bf)(m) := (-1)^{|f||b|} f(mb). \tag{2.1.13}$$

It is straightforward to check the associativity property. Similarly, for a graded A-module M and a graded (A,B)-bimodule N, the space $\mathrm{Hom}_A(M,N)$ is a graded right B-module, with multiplication given by

$$(fb)(m) = f(m)b. \tag{2.1.14}$$

Moreover, for a graded (A,B)-bimodule M and a graded B-module N, the graded tensor product $M \otimes_B N$ over B is graded A-module, with multiplication map $A \otimes (M \otimes_B N) \to M \otimes_B N$ given by

$$a(m \otimes n) = (am) \otimes n, \tag{2.1.15}$$

for homogeneous $a \in A$, $m \in M$, $n \in N$.

If the graded homomorphisms or graded tensor products are 'external', i.e. defined over the base field rather than a graded algebra, it is possible to transfer all the operators in a compatible way. For example, for a graded A-module M and a graded B-module N, the space $\mathrm{Hom}(M,N)$ of k-linear graded maps $f\colon M_\bullet \to N_\bullet$ is a graded (B,A)-bimodule, with multiplication

$$((b \otimes a^{\mathrm{op}})f)(m) = (bfa)(m) := (-1)^{|f||a|} b(f(am)), \tag{2.1.16}$$

for homogeneous $a \in A$, $b \in B$, $m \in M$, whereas for a graded right A-module M and a graded right B-module N, the space $\mathrm{Hom}(M,N)$ of graded k-linear maps $f\colon M \to N$ is a graded (A,B)-bimodule, with multiplication

$$\left((a \otimes b^{\mathrm{op}})f\right)(m) = (afb)(m) := (-1)^{|f|(|a|+|b|)} f(ma)b. \tag{2.1.17}$$

Similarly, for a graded right A-module M and a graded B-module N, the space $\mathrm{Hom}(M,N)$ of k-linear maps $f\colon M \to N$ is a a graded $(A \otimes B)$-module, with multiplication given by

$$((a \otimes b)f)(m) = (-1)^{|f|(|a|+|b|)} bf(ma). \tag{2.1.18}$$

Finally, as a final example, the graded tensor product $M \otimes N$ over the base field k is a graded $(A \otimes B)$-module, with multiplication given by

$$(a \otimes b)(m \otimes n) := (-1)^{|b||m|} (am) \otimes (bn). \tag{2.1.19}$$

Graded modules underlying a graded algebra

The underlying graded vector space of a graded algebra A is automatically a graded A^{e}-module, denoted ${}_{A^{\mathrm{e}}}A$, with multiplication given by

$$(a \otimes b^{\mathrm{op}})m = (-1)^{|m||b|}amb,$$

for homogeneous $a \otimes b^{\mathrm{op}} \in A^{\mathrm{e}}$, $m \in {}_{A^{\mathrm{e}}}A$ or, in other words, a graded A-bimodule, denoted ${}_AA_A$, with the left and right multiplications am and ma given by multiplication in A, for homogeneous $a \in A$ and $m \in {}_AA_A$. Then ${}_AA$ and A_A respectively denote the graded left and right A-modules with underlying graded vector space A, with the graded A-module structures respectively given by left and right multiplications.

2.1.1 Outer and inner bimodule structures

By the construction discussed in the previous subsection, the underlying graded vector space of A^{e} becomes a graded $(A^{\mathrm{e}})^{\mathrm{e}}$-module ${}_{(A^{\mathrm{e}})^{\mathrm{e}}}(A^{\mathrm{e}})$, with multiplication

$$((a_1 \otimes b_1^{\mathrm{op}}) \otimes (a_2 \otimes b_2^{\mathrm{op}})^{\mathrm{op}})\,(a \otimes b^{\mathrm{op}}) = \pm((a_1aa_2) \otimes (b_2bb_1)^{\mathrm{op}})$$

for homogeneous $(a_1 \otimes b_1^{\mathrm{op}}) \otimes (a_2 \otimes b_2^{\mathrm{op}})^{\mathrm{op}} \in (A^{\mathrm{e}})^{\mathrm{e}}$, $a \otimes b^{\mathrm{op}} \in {}_{(A^{\mathrm{e}})^{\mathrm{e}}}(A^{\mathrm{e}})$ and where $\pm = (-1)^{(|a_2||a|+|b||b_1|+|b_1||a_2|+|b_1||b_2|)}$. Moreover, by the equivalent descriptions of §2.1, this graded A^{e}-bimodule ${}_{A^{\mathrm{e}}}(A^{\mathrm{e}})_{A^{\mathrm{e}}} = {}_{(A^{\mathrm{e}})^{\mathrm{e}}}(A^{\mathrm{e}})$ corresponds to a pair of commuting graded left and right A^{e}-module structures on A^{e}.

Let $(A \otimes A)_{\mathrm{out}}$ be the graded A-bimodule corresponding to the graded left A^{e}-module structure ${}_{A^{\mathrm{e}}}(A^{\mathrm{e}})$, and $(A \otimes A)_{\mathrm{inn}}$ the A-bimodule corresponding to the graded right A^{e}-module structure ${}_{(A^{\mathrm{e}})^{\mathrm{op}}}(A^{\mathrm{e}}) = (A^{\mathrm{e}})_{A^{\mathrm{e}}}$ via (2.1.11) and (2.1.12) applied to $M = {}_{(A^{\mathrm{e}})^{\mathrm{op}}}(A^{\mathrm{e}})$. In other words,

$$(A \otimes A)_{\mathrm{out}} := {}_{A^{\mathrm{e}}}A^{\mathrm{e}}, \quad (A \otimes A)_{\mathrm{inn}} := \tau^*(A^{\mathrm{e}}_{A^{\mathrm{e}}}). \tag{2.1.20}$$

In the second identity, we will often drop the symbol τ^* and use the symbol $A^{\mathrm{e}}_{A^{\mathrm{e}}}$ to indicate the graded right A^{e}-module structure. By (2.1.8), (2.1.11) and (2.1.12) applied to $m = a \otimes b$, these graded bimodule structures are given by

$$\begin{aligned} a_1(a \otimes b)b_1 &= (-1)^{(|a|+|b|)|b_1|}(a_1 \otimes b_1^{\mathrm{op}})(a \otimes b^{\mathrm{op}}) \\ &= (-1)^{|b||b_1|}(a_1a) \otimes (b_1^{\mathrm{op}}b^{\mathrm{op}}) \\ &= (a_1a) \otimes (bb_1) \text{ in } (A \otimes A)_{\mathrm{out}}, \end{aligned}$$

On the other hand, by (2.1.11),

$$\begin{aligned} b_2 * (a \otimes b) * a_2 &= (-1)^{|a_2|(|a|+|b_2|)}(b_2 \otimes a_2^{\mathrm{op}}) * (a \otimes b^{\mathrm{op}}) \\ &= (-1)^{|a_2|(|a|+|b_2|)}\tau(b_2 \otimes a_2^{\mathrm{op}})(a \otimes b^{\mathrm{op}})^{\mathrm{op}} \\ &= (-1)^{|a_2|(|a|+|b_2|)}(-1)^{|a_2||b_2|}(a_2 \otimes b_2^{\mathrm{op}})^{\mathrm{op}}(a \otimes b^{\mathrm{op}})^{\mathrm{op}} \\ &= (-1)^{(|a_2||b_2|+|a_2||b|+|b_2||a|)}(aa_2) \otimes (b_2b) \text{ in } (A \otimes A)_{\mathrm{inn}}, \end{aligned}$$

To sum up,

(2.1.21a) $$a_1(a\otimes b)b_1 = (a_1a)\otimes(bb_1) \text{ in } (A\otimes A)_{\mathrm{out}},$$

(2.1.21b) $$b_2 * (a\otimes b) * a_2 = (-1)^{(|a_2||b_2|+|a_2||b|+|b_2||a|)}(aa_2)\otimes(b_2b) \text{ in } (A\otimes A)_{\mathrm{inn}},$$

and so it is usual to call them the *outer* and the *inner* A-bimodule structures of $A\otimes A$.

Dual graded modules

The *graded dual* of a graded A-module M is the A^{op}-module

(2.1.22) $$M^\vee := \mathrm{Hom}_A(M, {}_AA),$$

where by (2.1.14) applied to $N = {}_AA_A$, the multiplication $A^{\mathrm{op}}\otimes M^\vee \to M^\vee$ is given by

(2.1.23) $$(a^{\mathrm{op}}f)(m) = (fa)(m) := f(m)a,$$

for homogeneous $f\in M^\vee$, $a^{\mathrm{op}}\in A^{\mathrm{op}}$, $m\in M$. Since $(A^{\mathrm{op}})^{\mathrm{op}} = A$ and ${}_{A^{\mathrm{op}}}A^{\mathrm{op}} = A_A$, this definition applied to A^{op} implies that the graded dual of a graded A^{op}-module N is the graded A-module

(2.1.24) $$N^\vee := \mathrm{Hom}_{A^{\mathrm{op}}}(N, {}_{A^{\mathrm{op}}}A^{\mathrm{op}}) = \mathrm{Hom}_A(N, A_A)$$

with multiplication determined by the graded left A-module structure ${}_AA$ of A.

Evaluation maps

Define canonical maps

(2.1.25a) $$\mathtt{dual}_M\colon M \longrightarrow M^{\vee\vee} = \mathrm{Hom}_A(M^\vee, A_A),$$

(2.1.25b) $$\mathtt{eval}_{M,N}\colon M^\vee\otimes_A N \longrightarrow \mathrm{Hom}_A(M,N),$$

for graded A-modules M and N, by

$$(\mathtt{dual}(m))(f) := f(m), \quad (\mathtt{eval}(f\otimes n))(m) := f(m)n,$$

for $f\in M^\vee$, $m\in M$, $n\in N$. These maps are graded A-module morphisms, they are natural in M and N and, furthermore, they are additive in the sense that

(2.1.26a) $$\mathtt{dual}_M = \mathtt{dual}_{M_1}\oplus\mathtt{dual}_{M_2}, \text{ if } M = M_1\oplus M_2,$$

(2.1.26b) $$\mathtt{eval}_{M,N} = \bigoplus_{i,j=1,2}\mathtt{eval}_{M_i,N_j}, \text{ if } M = M_1\oplus M_2,\ N = N_1\oplus N_2.$$

Next, for graded modules ${}_AM$, N_A, ${}_BM'$, ${}_BN'$, ${}_AM''_B$, ${}_BN''_A$, ${}_AP$ over graded algebras A, B and C, we also define maps

(2.1.27a) $$\varphi\colon \mathrm{Hom}_{A\otimes B}(M\otimes M', \mathrm{Hom}(N,N')) \longrightarrow \mathrm{Hom}(N\otimes_A M, \mathrm{Hom}_B(M',N')),$$

(2.1.27b) $$\psi\colon \mathrm{Hom}_A(M,N)\otimes\mathrm{Hom}_B(M',N') \longrightarrow \mathrm{Hom}_{A\otimes B}(M\otimes M', N\otimes N'),$$

(2.1.27c) $$\eta\colon \mathrm{Hom}_A(M''\otimes_B N'', P) \longrightarrow \mathrm{Hom}_B(M'', \mathrm{Hom}_A(N'',P)).$$

where the unadorned Hom spaces and tensor products are 'external', i.e. over the base field k, so $\mathrm{Hom}(N, N')$ is a graded $A \otimes B$-bimodule (as in (2.1.18)), whereas $M \otimes M'$ and $N \otimes N'$ are graded $(A \otimes B)$-modules (as in (2.1.19)). These maps are given by

$$\begin{aligned}((\varphi f)(n \otimes m))m' &= (f(m \otimes m'))n,\\ (\psi(g \otimes g'))(m \otimes m') &= (-1)^{|g'||m|} g(m) \otimes g'(m'),\\ (\eta h)(m'')(n'') &= h(m'' \otimes n'').\end{aligned}$$

for homogeneous $f \in \mathrm{Hom}_{A \otimes B}(M \otimes M', \mathrm{Hom}(N, N'))$, $g \in \mathrm{Hom}_A(M, N)$, $g' \in \mathrm{Hom}_B(M', N')$, $h \in \mathrm{Hom}_{A \otimes B}(M'' \otimes_C N'', P)$, $m \in M$, $n \in N$, $m' \in M'$, $n' \in N'$, $m'' \in M''$, $n'' \in N''$. Furthermore, it is easy to see that both φ and η are isomorphisms.

In the special case $B = k$, ${}_BM' = {}_kk$ and $M' = V$, the map (2.1.27b) becomes

$$\psi_l \colon \mathrm{Hom}_A(M, N) \otimes V \longrightarrow \mathrm{Hom}_A(M, N \otimes V), \tag{2.1.28}$$

for two graded A-modules M, N, and a graded k-vector space V, where

$$(\psi_l(g \otimes v))(m) = (-1)^{|v||m|} g(m) \otimes v,$$

for homogeneous $g \in \mathrm{Hom}_A(M, N)$, $v \in V$, $m \in M$. Similarly, we obtain

$$\psi_r \colon V \otimes \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(M, V \otimes N). \tag{2.1.29}$$

Graded Dual Bimodules

Applying the constructions of the previous subsection, with A replaced by A^{e}, for a graded A^{e}-module M, we obtain a graded right A^{e}-module

$$M^{\vee}_{A^{\mathrm{e}}} = \mathrm{Hom}_{A^{\mathrm{e}}}(M, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) = \mathrm{Hom}_{A^{\mathrm{e}}}(M, (A \otimes A)_{\mathrm{out}}), \tag{2.1.30}$$

whose elements are graded A-bimodule morphisms

$$f \colon M \longrightarrow {}_{A^{\mathrm{e}}}A^{\mathrm{e}} = (A \otimes A)_{\mathrm{out}},$$

that is, graded linear maps such that

$$f(a_1 m a_2) := (-1)^{|f||a_1|} a_1 f(m) a_2,$$

for homogeneous $m \in M$, $a_1, a_2 \in A$, and where the $(A^{\mathrm{e}})^{\mathrm{op}}$-module structure on $M^{\vee}$ is induced by the right A^{e}-module $A^{\mathrm{e}}_{A^{\mathrm{e}}}$. Converting the right A^{e}-module structures $A^{\mathrm{e}}_{A^{\mathrm{e}}}$ and $M^{\vee}_{A^{\mathrm{e}}}$ into A^{e}-module structures as in §2.1, we see that the inner bimodule structure $(A \otimes A)_{\mathrm{inn}} = \tau^*(A^{\mathrm{e}}_{A^{\mathrm{e}}})$ given by (2.1.21b) makes $\tau^*(M^{\vee})$ into a graded A-bimodule with multiplication

$$(a_1 * f * a_2)(m) := (-1)^{|f||a_1|} a_1 * f(m) * a_2, \tag{2.1.31}$$

for homogeneous $f \in M^\vee$, $a_1, a_2 \in A$ and $m \in M$, where we used (2.1.10), (2.1.14) and that the homogeneous element of f of $\mathrm{Hom}_A(N, A_A)$ corresponds to $f^{\mathrm{op}} \in \mathrm{Hom}_{A^{\mathrm{op}}}(N, {}_{A^{\mathrm{op}}}A^{\mathrm{op}})$ for a graded $(A^{\mathrm{e}})^{\mathrm{op}}$-module. We will often drop the symbol τ^* and use the symbols $M^\vee_{A^{\mathrm{e}}}$ and ${}_{A^{\mathrm{e}}}M^\vee := \tau^*(M^\vee)$ to indicate the graded right and left A^{e}-module structures, respectively.

Applying (2.1.10) to the graded right A^{e}-module $M^\vee_{A^{\mathrm{e}}}$, and the construction (2.1.22) to the graded A^{e}-module ${}_{A^{\mathrm{e}}}M^\vee = \tau^*(M^\vee)$, we obtain a graded A^{e}-module $M^{\vee\vee} := (M^\vee_{A^{\mathrm{e}}})^\vee$ and a graded right A^{e}-module $(\tau^* M^\vee)^\vee := ({}_{A^{\mathrm{e}}}(\tau^*(M^\vee)))^\vee$, respectively, where the elements of $M^{\vee\vee}$ are graded linear maps $\phi\colon M^\vee \to (A \otimes A)_{\mathrm{inn}}$ such that

$$\phi(a_1 * f * a_2) := (-1)^{(|\phi|+|f|)|a_1|} a_1 * \phi(f) * a_2,$$

for all homogeneous $f \in M^\vee$, $a_1, a_2 \in A$, and the elements of $(\tau^* M^\vee)^\vee$ are graded linear maps $\phi'\colon M^\vee \to (A \otimes A)_{\mathrm{out}}$ such that

$$\phi(a_1 f a_2) := (-1)^{(|\phi|+|f|)|a_1|} a_1 \phi(f) a_2,$$

for all homogeneous $f \in M^\vee$, $a_1, a_2 \in A$. Then the graded A^{e}-bimodule structure on $M^{\vee\vee}$ is

$$(a_1 \phi a_2)(f) = (-1)^{|\phi||a_1|} a_1 \phi(f) a_2,$$

and the graded $(A^{\mathrm{e}})^{\mathrm{op}}$-bimodule structure of $\tau^*(\tau^* M^\vee)^\vee$ is

$$(a_1 * \phi' * a_2)(f) = (-1)^{|\phi||a_1|} a_1 * \phi'(f) * a_2.$$

Lemma 2.1.32. *τ induces a graded A^{e}-module isomorphism*

$$\tau_*\colon M^{\vee\vee} \xrightarrow{\cong} \tau^*(\tau^* M^\vee)^\vee\colon \quad \phi \longmapsto \tau \circ \phi.$$

Note that if the ungraded underlying A^{e}-module corresponding to the graded A^{e}-module $M^{\vee\vee}$ is finitely generated and projective, then so is the underlying ungraded of $\tau^*(\tau^* M^\vee)^\vee \cong M^{\vee\vee}$, and hence the underlying ungraded of the graded $(A^{\mathrm{e}})^{\mathrm{op}}$-module $(\tau^* M^\vee)^\vee$ is also finitely generated and projective.

Finally, by the construction of (2.1.25a), with A replaced by A^{e}, we obtain an additive graded A-bimodule morphism

$$\texttt{bidual}_M\colon M \longrightarrow M^{\vee\vee} = \mathrm{Hom}_A(M^\vee, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) = \mathrm{Hom}_A(M^\vee, (A \otimes A)_{\mathrm{out}}), \tag{2.1.33}$$

natural in the graded A-bimodule M, where

$$(\texttt{bidual}(m))(f) := f(m)$$

for homogeneous $f \in M^\vee$ and $m \in M$.

2.1.2 Finitely generated projective modules

We will now collect the following results about finitely generated projective modules over an associative algebra A.

(i) If M is a finitely generated and projective A-module, then $M^\vee$ is a finitely generated projective right A-module and the map

$$\mathtt{dual}_M \colon M \longrightarrow M^{\vee\vee}$$

of (2.1.25a) is an isomorphism of A-modules.

(ii) If M is a finitely generated projective A-module, then the map

$$\mathtt{eval}_{M,N} \colon M^\vee \otimes_A N \longrightarrow \operatorname{Hom}_A(M, N)$$

of (2.1.25b) is an isomorphism, for any A-module N.

(iii) For modules ${}_AM$, N_A, ${}_BM'$ and ${}_BN'$ over algebras A and B, the map ψ of (2.1.27b) is an isomorphism, if M is a finitely generated projective A-module and M' is a finitely generated projective B-module.

(iv) For modules ${}_AM$ and ${}_AN$ and a vector space V (over k), the map

$$\psi_l \colon \operatorname{Hom}_A(M, N) \otimes V \longrightarrow \operatorname{Hom}_A(M, N \otimes V)$$

in (2.1.28) is an isomorphism, provided that M is a finitely generated projective A-module. Similarly, if M is a finitely generated projective A-module, then the map

$$\psi_r \colon V \otimes \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, V \otimes N)$$

of (2.1.29) is an isomorphism.

For part (iv), note that it is a special case of part (iii), corresponding to two A-modules M and N, for an algebra A, and a vector space V, because, in this case, the map ψ_l of (2.1.27b) becomes the map $\psi\colon \operatorname{Hom}_A(M,N) \otimes V \to \operatorname{Hom}_A(M, N \otimes V)$ of (2.1.28).

Replacing now A by A^{e}, we obtain similar results for A^{e}-modules and $(A^{\mathrm{e}})^{\mathrm{op}}$-modules. In particular,

(v) If M is a finitely generated projective A^{e}-module, then $M^\vee$ is a finitely generated projective right A^{e}-module and the map $\mathtt{bidual}_M \colon M \to M^{\vee\vee}$ of (2.1.33) is an isomorphism of A^{e}-modules.

The proofs of these results are well-known and they can be found in a lot of references (see, for instance, [64] or [66]). Finally, these formulae will be used throughout in this thesis for the underlying ungraded modules of graded modules over graded algebras.

2.2 Basics on Noncommutative Algebraic Geometry

2.2.1 (Graded) Non-commutative differential 1-forms

In the rest of this chapter, we fix the following framework: let R be an associative k-algebra over a field of characteristic zero k, and A be a graded R-algebra, i.e. a graded algebra together with a graded algebra homomorphism $R \to A$. Given a graded A-bimodule M, a *derivation of weight* n of A into M is an additive map $\theta\colon A \to M$ satisfying the Leibniz rule (see (2.3.10)):

$$\theta(ab) = (-1)^{n|a|}\theta(a)b + a\theta(b), \tag{2.2.1}$$

for all homogeneous $a, b \in A$, and an *R-linear derivation of weight* n of A into M is a derivation of weight n, $\theta\colon A \to M$, which is a morphism $\theta\colon {}_RA_R \to {}_RM_R$ of graded R-bimodules (equivalently, $\theta(R) = 0$).

Based on the arguments introduced in §2.1, the space $\mathrm{Der}^n_R(A, M)$ of R-linear derivations of weight n of A into M is a $Z(A^{\mathrm{e}})$-module. Furthermore,

$$\mathrm{Der}_R(A, M) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}^n_R(A, M)$$

is a graded $Z(A^{\mathrm{e}})$-module shall be called the *space of graded derivations of A into M*. Note that, in particular, it is a graded k-module, because the image of k under the structure map $k \to A^e$ is in $Z(A^{\mathrm{e}})$.

Following [31], Section 2, and [75] for the graded setting, we define the *graded bimodule of non-commutative differential 1-forms* as the graded A-bimodule $\Omega^1_R A$ endowed with an R-derivation of weight zero,

$$\mathrm{d}\colon A \longrightarrow \Omega^1_R A\colon \quad a \longmapsto \mathrm{d}a, \tag{2.2.2}$$

which satisfies the following universal property: for every graded A-bimodule M and R-linear derivation $\theta\colon A \to M$ of weight $|\theta|$, there exists a unique morphism of graded A-bimodules $i_\theta\colon \Omega^1_R A \to M$ such that $\theta = i_\theta \circ d$. In other words, the graded A-bimodule $\Omega^1_R A$ represents the functor $\mathrm{Der}_R(A, -)$ from the category of graded A-bimodules into the category of graded k-modules[1] It is worthwhile to observe that $|i_\theta| = |\theta|$.

Hence, by the universal property, there exists a canonical isomorphism of graded A-bimodules,

$$\mathrm{Der}_R(A, M) \xrightarrow{\cong} \mathrm{Hom}_{A^e}(\Omega^1_R A, M)\colon \quad \theta \longmapsto i_\theta, \tag{2.2.3}$$

[1] Explicitly, the bimodule $\Omega^1_R A$ is generated over A by the set of symbols $\{\mathrm{d}a \mid a \in A\}$ under the relations

(i) $\mathrm{d}(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1\, \mathrm{d}a_1 + \lambda_2\, \mathrm{d}a_2$;

(ii) $\mathrm{d}(a_1 a_2) = a_1(\mathrm{d}a_2) + (-1)^{|a|}(\mathrm{d}a_1)a_2$,

for all $\lambda_1, \lambda_2 \in k$.

whose inverse map is given by $i_\theta \mapsto \theta = i_\theta \circ \mathrm{d}$. In particular, since i_θ is a graded A^{e}-module morphism, $i_\theta(\overline{a}\,\mathrm{d}a) = (-1)^{|\theta||\overline{a}|}\overline{a}\; i_\theta(\mathrm{d}a) = (-1)^{|\theta||\overline{a}|}\overline{a}\theta(a)$, for homogeneous $\overline{a} \in A^{\mathrm{e}}$, $a \in A$.

2.2.2 Graded Double Derivations and Non-commutative differential forms

Let

$$\mathbb{D}\mathrm{er}_R\, A := \mathrm{Der}_R(A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) = \mathrm{Der}_R(A, (A \otimes A)_{\mathrm{out}}) \tag{2.2.4}$$

be the graded A^{e}-module of *double derivations*, whose graded A-bimodule structure also comes via the map (2.1.5) from $A^{\mathrm{e}}_{A^{\mathrm{e}}}$, or in other words, from the inner graded A-bimodule structure (2.1.21b), i.e.,

$$(b * \Theta * c)(a) = (-1)^{|b||\Theta'(a)|+|b||c|+|c||\Theta''(a)|}\Theta'(a)c \otimes b\Theta''(a),$$

for all homogeneous $\Theta \in \mathbb{D}\mathrm{er}_R\, A$, $b \otimes c^{\mathrm{op}} \in A^{\mathrm{e}}$ and $a \in A$. Applying the functor $\mathrm{Hom}_{A^{\mathrm{e}}}(-{}_{A^{\mathrm{e}}}A^{\mathrm{e}})$ to $\Omega^1_R A$, we obtain a right graded A^{e}-module

$$(\Omega^1_R A)^\vee = \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) = \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, (A \otimes A)_{\mathrm{out}}), \tag{2.2.5}$$

which can be regarded as a graded A-bimodule and, as in §2.1.1, the graded A-bimodule structure comes via the map (2.1.5) (see (2.1.10)) from $A^{\mathrm{e}}_{A^{\mathrm{e}}}$, or in other words, from the inner graded A-bimodule structure (2.1.21b).

By the universal property (2.2.3) of $\Omega^1_R A$, (2.2.4) and (2.2.5), we have the canonical isomorphism:

$$\begin{aligned} \texttt{canonical}\colon \mathbb{D}\mathrm{er}_R\, A &\xrightarrow{\cong} (\Omega^1_R A)^\vee \\ \Theta &\longmapsto i_\Theta, \end{aligned} \tag{2.2.6}$$

where

$$i_\Theta\colon \Omega^1_R A \longrightarrow A \otimes A\colon \quad \alpha \longmapsto (-1)^{|i'_\Theta(\alpha)||i''_\Theta(\alpha)|}\sigma_{(12)} i_\Theta \alpha.$$

Observe that (2.2.6) is a graded A^{e}-module morphism because we are using the permutation $\sigma_{(12)}\colon A \otimes A \to A \otimes A\colon\ a_1 \otimes a_2 \mapsto a_2 \otimes a_1$. The algebra of *noncommutative differential forms* of A is the tensor algebra

$$\Omega^\bullet_R A := \mathrm{T}^\bullet_A\, \Omega^1_R A \tag{2.2.7}$$

of the (graded) A-bimodule $\Omega^1_R A$ if $n \geq 0$ whereas $\Omega^n_R A = 0$ if $n < 0$. Finally, by convention, $\Omega^0_R A = A$.

Since A is a graded R-algebra, $\mathrm{T}_A\, \Omega^1_R A$ is a bi-complex. The grading by 'form degree' will be denoted by $\|-\|$ (and called the *degree*), whereas the grading induced from the A-grading shall be denoted by $|{-}|$ (and called the *weight*). It is important to observe that they interact by means of the so-called *Koszul sign* *convention*, whereby the sign associated to two homogeneous elements $a, b \in A$ of bi-degrees $(\|a\|, |a|)$ and $(\|b\|, |b|)$ is $(-1)^{(\|a\|,|a|)\cdot(\|b\|,|b|)}$, where

$$(\|a\| \; |a|) \cdot (\|b\|, |b|) := \|a\| \cdot \|b\| + |a| \cdot |b|. \tag{2.2.8}$$

Since $\Omega^\bullet_R$ is a differential bi-graded algebra over A, we shall extend the differential d in (2.2.2) to a derivation in $\Omega^\bullet_R$ of weight 0 and degree +1 using the Leibniz rule. Moreover, as $\Omega^\bullet_R A$ is the free algebra of the graded bimodule $\Omega^1_R A$, there exists a unique extension of the map $i_\Theta \colon \Omega^1_R A \to A \otimes A$ to a double derivation of bi-degree $(-1, |\Theta|)$ on $\mathrm{T}_A\, \Omega^1_R A$ with respect to the bigrading $(\|-\|, |-|)$,

$$i_\Theta \, : \Omega^\bullet_R A \longrightarrow \bigoplus (\Omega^i_R A \otimes \Omega^j_R A), \tag{2.2.9}$$

where the sum is over pairs (i, j) with $i + j = \bullet - 1$.

As we wrote above, in particular, i_Θ is a double derivation for a homogeneous $\Theta \in \mathbb{D}\mathrm{er}_R\, A$ and satisfies the following Leibniz rule:

$$\begin{aligned} i_\Theta(\alpha\beta) &= (-1)^{|\Theta||\alpha|}(i_\Theta\alpha)\beta + (-1)^{\|\alpha\|\,\|i_\Theta\|}\alpha(i_\Theta\beta) \\ &= (-1)^{|\Theta||\alpha|}(i_\Theta\alpha)\beta + (-1)^{\|\alpha\|}\alpha(i_\Theta\beta). \end{aligned} \tag{2.2.10}$$

Observe that the map i_Θ should be regarded as a super-derivation of the graded algebra $\Omega^\bullet_R A$ with coefficients in $\Omega^\bullet_R A \otimes \Omega^\bullet_R A$, viewed as an $\Omega^\bullet_R A$-bimodule with respect to the outer bimodule structure. In particular, if $\mathrm{d}a, \mathrm{d}b \in \Omega^1_R A$, by (2.2.10),

$$\begin{aligned} i_\Theta(\mathrm{d}a\, \mathrm{d}b) &= (-1)^{|\Theta||a|} i_\Theta(\mathrm{d}a)\, \mathrm{d}b - \mathrm{d}a i_\Theta(\mathrm{d}b) \\ &= (-1)^{|\Theta||a|}(\Theta'(a) \otimes \Theta''(a))(1 \otimes \mathrm{d}b) - (\mathrm{d}a \otimes 1)(\Theta'(b) \otimes \Theta''(b) \\ &= (-1)^{|\Theta||a|}\Theta'(a) \otimes (\Theta''(a)\, \mathrm{d}b) - (\mathrm{d}a\Theta'(b)) \otimes \Theta''(b). \end{aligned} \tag{2.2.11}$$

Finally, we point out that for every homogeneous element $a \in A$, we have $i_\Theta(a) = 0$ if $\Theta \in \mathbb{D}\mathrm{er}_R\, A$ because $|i_\Theta| = -1$. The *Lie derivative* with respect to $\Theta \in \mathbb{D}\mathrm{er}_R\, A$ is the map

$$L_\Theta \colon \Omega^1_R A \longrightarrow (A \otimes \Omega^1_R A) \oplus (\Omega^1_R A \otimes A)$$

defined on generators by $L_\Theta(a) = \Theta(a)$ and $L_\Theta(\mathrm{d}a) = \mathrm{d}\Theta(a)$, for homogeneous $a \in A$, and we extend it to $\Omega^1_R A$ by imposing the Leibniz rule. By a simple calculation on the generators, one obtains the Cartan formula in this setting (see [30, (2.7.2)]):

Lemma 2.2.12.

$$L_\Theta = \mathrm{d} \circ i_\Theta + i_\Theta \circ \mathrm{d}.$$

Moreover, the Lie derivative L_Θ extends to a degree 0 derivation and weight $|\Theta|$,

$$L_\Theta \colon \; \Omega^\bullet_R A \to \bigoplus \left(\Omega^i_R A \otimes \Omega^j_R A\right) \subset (\Omega^\bullet_R A)^{\otimes 2}, \tag{2.2.13}$$

where the sum is now for pairs (i,j) with $i+j=\bullet$. A key technical point in non-commutative algebraic geometry is the fact that for every associative k-algebra A, the bi-complex $(\Omega_k^\bullet A, \mathrm{d}\,)$ has trivial cohomology:

$$H^i(\Omega_k^\bullet A, \mathrm{d}\,) = \begin{cases} k & \text{if } i=0, \\ 0 & \text{if } i>0. \end{cases}$$

As it is explained in [41], §11.4, we have an isomorphism $\Omega_k^p A = A \otimes (A/k)^{\otimes p}$, and the differential d corresponds to the natural projection

$$A \otimes (A/k)^{\otimes p} \longrightarrow (A/k) \otimes (A/k)^{\otimes p} \simeq k \otimes (A/k)^{\otimes (p+1)} \subset A \otimes (A/k)^{\otimes (p+1)}$$

The kernel of the latter projection is equal to $k \otimes (A/k)^{\otimes p}$, which is exactly the image of the differential d: $A \otimes (A/k)^{\otimes (p-1)} \longrightarrow k \otimes (A/k)^{\otimes p}$. To obtain an interesting cohomology, we consider the commutator quotient of the complex $(\Omega_R^\bullet A, \mathrm{d}\,)$. Let A be a graded R-algebra and define the *non-commutative Karoubi-de Rham bi-complex of* A as the bi-graded vector space

$$\mathrm{DR}_R^\bullet(A) := \Omega_R^\bullet A / [\Omega_R^\bullet A, \Omega_R^\bullet A],$$

where if $\omega, \omega' \in \Omega_R^\bullet A$, their bi-graded commutator is defined as (see [98])

$$[\omega, \omega'] := \omega\omega' - (-1)^{(\|\omega\|\|\omega'\| + |\omega||\omega'|)} \omega'\omega,$$

where we denote the bi-gradation of $\mathrm{DR}_R^\bullet(A)$ as in $\Omega_R^\bullet A$ and we use the Koszul convention (2.2.8). Finally, the differential $\mathrm{d}: \Omega_R^\bullet A \to \Omega_R^{\bullet+1} A$ descends to a well-defined differential $\mathrm{d}: \mathrm{DR}_R^\bullet(A) \to \mathrm{DR}_R^{\bullet+1}(A)$ and hence the Karoubi-de Rham bi-complex becomes a differential bi-graded vector space.

Given a graded R-algebra C and $c = c_1 \otimes c_2$, with $c_i \in C$, define

$$c^\circ := (-1)^{|c_1||c_2|} c_2 \otimes c_1,$$

and given a linear map $\phi\colon C \longrightarrow C^{\otimes 2}$, write

$$\phi^\circ\colon C \longrightarrow C\colon \quad c \longmapsto (\phi(c))^\circ.$$

In our case, $C = \Omega_R^\bullet A$ which is bi-graded, hence we have to deal with the weight inherited from A and the degree as differential form. Thus, if $\Theta \in \mathbb{D}\mathrm{er}_R A$,

$$\begin{aligned} i_\Theta^\circ\colon \Omega_R^\bullet A &\longrightarrow \Omega_R^\bullet A \otimes \Omega_R^\bullet A \\ \alpha &\longmapsto i_\Theta^\circ(\alpha) := (-1)^{\|i'_\Theta(\alpha)\|\|i''_\Theta(\alpha)\| + |i'_\Theta(\alpha)||i''_\Theta(\alpha)|} i''_\Theta(\alpha) \otimes i'_\Theta(\alpha) \end{aligned} \tag{2.2.14}$$

Moreover, define an operation between non-commutative differential forms:

$$\begin{aligned} m\colon \Omega_R^\bullet A \otimes \Omega_R^\bullet A &\longrightarrow \Omega_R^\bullet A \\ \alpha \otimes \beta &\longmapsto m(\alpha \otimes \beta) := \alpha\beta \end{aligned} \tag{2.2.15}$$

Combining (2.2.14) and (2.2.15), we are able to define the desired operator; the *reduced contraction operator* for a homogeneous $\Theta \in \mathbb{D}\mathrm{er}_R\, A$ and a bi-homogeneous $\alpha \in \Omega^\bullet_R A$ defined as

$$\begin{aligned} \iota_\Theta \colon \Omega^\bullet_R A &\longrightarrow \Omega^\bullet_R A \\ \alpha &\longmapsto \iota_\Theta \alpha, \end{aligned} \tag{2.2.16}$$

where

$$\iota_\Theta \alpha := (-1)^{\|i'_\Theta(\alpha)\|\|i''_\Theta(\alpha)\|+|i'_\Theta(\alpha)||i''_\Theta(\alpha)|} i''_\Theta(\alpha) i'_\Theta(\alpha). \tag{2.2.17}$$

Similarly, we define the *reduced Lie derivative*

$$\mathcal{L}_\Theta := {}^\circ L_\Theta \colon\ \Omega^\bullet_R A \longrightarrow \Omega^\bullet_R A. \tag{2.2.18}$$

We apply the operation ${}^\circ(-)$ to the Cartan formula in Lemma 2.2.12, obtaining the *reduced Cartan identity* which relates $\mathcal{L}_\Theta$ and ι_Θ:

Lemma 2.2.19 ([30], Lemma 2.8.8)**.** *For every homogenous $\Theta \in \mathbb{D}\mathrm{er}_R\, A$,*

$$\mathcal{L}_\Theta = \mathrm{d} \circ \iota_\Theta + \iota_\Theta \circ \mathrm{d}.$$

2.2.3 Smoothness

In this subsection we shall introduce the concept of smoothness which will be used throughout in this thesis. Recall (see [72]) that an associative algebra A is called *formally smooth* if the following lifting property holds: for every algebra B and a nilpotent two-sided ideal $I \subset B$ (that is, $I = BIB$ and $I^n = 0$ for $n \gg 0$), given a map $A \to B/I$, there is a lift $A \to B$ such that the following diagram commutes

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & B/I \end{tikzcd} \tag{2.2.20}$$

where $B \to B/I$ is the quotient map. The definition of formal smoothness via the lifting property (2.2.20) is analogous to Grothendieck's definition of formally smooth algebras in the commutative case.

From now on, let R be an associative k-algebra over a field of characteristic zero and A is an R-algebra. Then to obtain a notion of nonsingularity for noncommutative algebras, W. F. Schelter proved in [84] (see also [31], Proposition 3.2) the following result:

Proposition 2.2.21 ([84], Lemma 2.3)**.** *An R-algebra A is formally smooth if and only if the A-bimodule $\Omega^1_R A$ is projective.*

Motivated by this result, following [31] and [31], we make the following

Definition 2.2.22. An R-algebra A is called *smooth over* R if it is finitely generated as an R-algebra and $\Omega^1_R A$ is projective as an A^{e}-module.

Proposition 2.2.23. *If A is smooth over R and M is a finitely generated and projective A-bimodule, the algebra $\mathcal{A} = \mathrm{T}_A M$ is also smooth over R.*

Proof. [31], Proposition 5.3(3). □

2.2.4 The (iso)morphism $\mathtt{bidual}_{\Omega^1_R A}$

Let A be a graded smooth R-algebra. Applying (v) in §2.1.2 to $M = \Omega^1_R A$, we obtain the following isomorphism of graded A-bimodules:

$$\begin{aligned} \mathtt{bidual}_{\Omega^1_R A} \colon \Omega^1_R A &\xrightarrow{\cong} (\Omega^1_R A)^{\vee\vee} \\ \alpha &\longmapsto \alpha^{\vee}, \end{aligned} \tag{2.2.24}$$

where

$$(\Omega^1_R A)^{\vee\vee} = \mathrm{Hom}_A((\Omega^1_R A)^{\vee}, A^{\mathrm{e}}_{A^{\mathrm{e}}}) = \mathrm{Hom}_A((\Omega^1_R A)^{\vee}, (A \otimes A)_{\mathrm{inn}}),$$

and $\alpha^{\vee}(f) = f(\alpha)$ for homogeneous $f \in (\Omega^1_R A)^{\vee}$. Dualizing now the isomorphism (2.2.6), we obtain

$$(\mathtt{canonical})^{\vee} \colon (\Omega^1_R A)^{\vee\vee} \xrightarrow{\cong} (\mathbb{D}\mathrm{er}_R\, A)^{\vee} \colon \quad \varphi \mapsto i^{\vee}(\varphi), \tag{2.2.25}$$

where

$$\begin{aligned} i^{\vee}(\varphi) \colon \mathbb{D}\mathrm{er}_R\, A &\longrightarrow A \otimes A \\ \Theta &\longmapsto i^{\vee}(\varphi)(\Theta) = \varphi(i_{\Theta}). \end{aligned}$$

Composing the isomorphisms (2.2.24) and (2.2.25), we obtain the following isomorphism

$$\begin{aligned} \mathtt{Bidual}' \colon \Omega^1_R A &\xrightarrow{\cong} (\mathbb{D}\mathrm{er}_R\, A)^{\vee} \\ \alpha &\longmapsto i^{\vee}(\alpha^{\vee}), \end{aligned} \tag{2.2.26}$$

where

$$i^{\vee}(\alpha^{\vee}) \colon \mathbb{D}\mathrm{er}_R\, A \longrightarrow A \otimes A \colon \quad \Theta \longmapsto (-1)^{|i'_{\Theta}(\alpha)||i''_{\Theta}(\alpha)|} \sigma_{(12)} i_{\Theta}\alpha$$

By part (iv) of §2.1.2 applied to the graded algebra A^{e}, given a graded A-bimodule M and a graded vector space V, we obtain an isomorphism

$$\psi_l \colon \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, M) \otimes V \xrightarrow{\cong} \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, M \otimes V), \tag{2.2.27}$$

where the graded A-bimodule structure on the 'external' tensor product $M \otimes V$ comes from the graded A-bimodule M, i.e. the multiplication is given by $(a_1 \otimes a_2^{\mathrm{op}})(m \otimes v) = ((a_1 \otimes a_2^{\mathrm{op}})m) \otimes v$ for homogeneous $a_1 \otimes a_2^{\mathrm{op}} \in A^{\mathrm{e}}$, $m \in M$,

$v \in V$. This is used in [96] when $M = {}_{A^e}A^e = (A \otimes A)_{\mathrm{out}}$ and $V = A$. In this case, we have an isomorphism of graded A-bimodules given by

$$\begin{aligned} \tau_{(23)} \colon {}_{A^e}A^e \otimes A &\xrightarrow{\cong} A^{\otimes 3} \\ (a_1 \otimes a_2) \otimes a_3 &\longmapsto (-1)^{|a_2||a_3|} a_1 \otimes a_3 \otimes a_2, \end{aligned} \tag{2.2.28}$$

where in the right-hand side, $A^{\otimes 3}$ has its outer graded A-bimodule structure. This isomorphism induces another one, that will also be denoted $\tau_{(23)}$:

$$\tau_{(23)} \colon \operatorname{Hom}_{A^e}(\Omega^1_R A, {}_{A^e}A^e \otimes A) \xrightarrow{\cong} \operatorname{Hom}_{A^e}(\Omega^1_R A, A^{\otimes 3}) \xrightarrow{\cong} \operatorname{Der}_R(A, A^{\otimes 3}) \tag{2.2.29}$$

Using (2.2.27) for $M = {}_{A^e}A^e$ and $V = A$, (2.2.28) and (2.2.29),

$$\begin{aligned} \mathbb{D}\mathrm{er}_R\, A \otimes A &= \operatorname{Hom}_{A^e}(\Omega^1_R A, {}_{A^e}A^e) \otimes A \\ &\cong \operatorname{Hom}_{A^e}(\Omega^1_R A, {}_{A^e}A^e \otimes A) \\ &\cong \operatorname{Hom}_{A^e}(\Omega^1_R A, A^{\otimes 3}) \\ &\cong \operatorname{Der}_R(A, A^{\otimes 3}). \end{aligned} \tag{2.2.30}$$

Similarly, under the same assumptions on A, M and V, and by part (iv) in §2.1.2, we obtain an isomorphism

$$\psi_r \colon V \otimes \operatorname{Hom}_{A^e}(\Omega^1_R A, M) \xrightarrow{\cong} \operatorname{Hom}_{A^e}(\Omega^1_R A, V \otimes M). \tag{2.2.31}$$

Taking $M = {}_{A^e}A^e = (A \otimes A)_{\mathrm{out}}$ and $V = A$, we have the following isomorphism of graded A-bimodules:

$$\begin{aligned} \tau_{(12)} \colon A \otimes {}_{A^e}A^e &\xrightarrow{\cong} A^{\otimes 3} \\ a_1 \otimes (a_2 \otimes a_3) &\longmapsto (-1)^{|a_1||a_2|} a_2 \otimes a_1 \otimes a_3, \end{aligned} \tag{2.2.32}$$

where in the right-hand side, $A^{\otimes 3}$ has its outer graded A-bimodule structure. This isomorphism induces another one, that will also be denoted $\tau_{(12)}$:

$$\tau_{(12)} \colon \operatorname{Hom}_{A^e}(\Omega^1_R A, {}_{A^e}A^e) \xrightarrow{\cong} \operatorname{Hom}_{A^e}(\Omega^1_R A, A^{\otimes 3}) \xrightarrow{\cong} \operatorname{Der}_R(A, A^{\otimes 3}). \tag{2.2.33}$$

Now, applying the isomorphisms (2.2.31) for $M = {}_{A^e}A^e$ and $V = A$, (2.2.32) and (2.2.33):

$$\begin{aligned} A \otimes \mathbb{D}\mathrm{er}_R\, A &= A \otimes \operatorname{Hom}_{A^e}(\Omega^1_R A, {}_{A^e}A^e) \\ &\cong \operatorname{Hom}_{A^e}(\Omega^1_R A, A \otimes {}_{A^e}A^e) \\ &\cong \operatorname{Hom}_{A^e}(\Omega^1_R A, A^{\otimes 3}) \\ &\cong \operatorname{Der}_R(A, A^{\otimes 3}) \end{aligned} \tag{2.2.34}$$

2.3 Graded double Poisson structures

2.3.1 Double Poisson algebras of weight $-N$

Commutative Poisson algebras appear in several geometric and algebraic contexts. As a direct non-commutative generalization, one might consider associative algebras that are at the same time Lie algebras under a ‘Poisson bracket’ $\{-,-\}$

satisfying the Leibniz rules $\{ab,c\} = a\{b,c\} + \{a,c\}b$, $\{a,bc\} = b\{a,c\} + \{a,b\}c$. However such Poisson brackets are the commutator brackets up to a scalar multiple, provided the algebra is prime and not commutative [36, Theorem 1.2]. A way to resolve this apparent lack of noncommutative Poisson algebras is provided by double Poisson structures, introduced by M. Van den Bergh in [96], §2.2.

An n-*bracket* on an associative algebra A is a linear map

$$\{\!\{-,\cdots,-\}\!\} : A^{\otimes n} \longrightarrow A^{\otimes n},$$

which is a derivation $A \to A^{\otimes n}$ in its last argument for the outer bimodule structure on $A^{\otimes n}$, i.e.

$$\{\!\{a_1,a_2,...,a_{n-1}a_n'\}\!\} = a_{n-1}\{\!\{a_1,a_2,...,a_n'\}\!\} + \{\!\{a_1,a_2,...,a_{n-1}\}\!\}\, a_n'$$

and which is cyclically anti-symmetric in the sense

$$\tau_{(1\cdots n)} \circ \{\!\{-,\cdots,-\}\!\} \circ \tau^{-1}_{(1\cdots n)} = (-1)^{n+1}\{\!\{-,\cdots,-\}\!\}.$$

If A is an R-algebra, then an n-bracket is R-*linear* if it vanishes when its last argument is in the image of R. Observe that a 1-bracket is a derivation. Next, let $\{\!\{-,-\}\!\}$ be a double bracket on A, $a \in A$, and $b = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$. Then we define

$$\begin{aligned}\{\!\{a,b\}\!\}_L &= \{\!\{a,b_1\}\!\} \otimes b_2 \otimes \cdots \otimes b_n,\\ \{\!\{a,b\}\!\}_R &= b_1 \otimes \cdots \otimes b_{n-1} \otimes \{\!\{a,b_n\}\!\},\end{aligned}$$

which enables us to introduce an appropriate analogue of the Jacobi identity in this setting:

Definition 2.3.1 (Double Poisson algebra [96])**.** A *double bracket* on A is a linear map

$$\{\!\{-,-\}\!\} : A \otimes A \longrightarrow A \otimes A,$$

which satisfies, for all $a,b,c \in A$,

$$\{\!\{a,b\}\!\} = -\sigma_{(12)}\{\!\{b,a\}\!\}, \tag{2.3.2a}$$

$$\{\!\{a,bc\}\!\} = b\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}c. \tag{2.3.2b}$$

Furthermore, a double bracket $\{\!\{-,-\}\!\}$ on A is a *double Poisson bracket* if satisfies the *double Jacobi identity*:

$$0 = \{\!\{a,\{\!\{b,c\}\!\}\}\!\}_L + \sigma_{(123)}\{\!\{b,\{\!\{c,a\}\!\}\}\!\}_L + \sigma_{(132)}\{\!\{c,\{\!\{a,b\}\!\}\}\!\}_L. \tag{2.3.3}$$

An algebra with a double Poisson bracket $(A, \{\!\{-,-\}\!\})$ is a *double Poisson algebra.*

Remark 2.3.4. Observe that (2.3.2a) can be rewritten as

$$\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^\circ,$$

where we used the operator $(-)^\circ$ introduced in §2.2.2.

Moreover, note that the formulas (2.3.2a) and (2.3.2b) imply that $\{\!\{-,-\}\!\}$ is a derivation $A \to A \otimes A$ in its first argument for the inner bimodule structure on $A \otimes A$, that is,

$$\{\!\{ab, c\}\!\} = a * \{\!\{b, c\}\!\} + \{\!\{a, c\}\!\} * b,$$

where, as usual, by $*$ we mean the inner action and $a, b \in A$.

If $\{\!\{-,-\}\!\}$ is a double bracket, M. Van den Bergh defined in [96], §2.4, the *bracket associated to* $\{\!\{-,-\}\!\}$:

$$\{-,-\}\colon A \otimes A \longrightarrow A\colon \quad (a,b) \longmapsto \{a,b\} := m \circ \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}'' \tag{2.3.5}$$

Here m denotes the multiplication. It is clear that $\{-,-\}$ is a derivation in its second argument. Recall (see [?]) that a *left Loday algebra* is a vector space V equipped with a bilinear operation $[-,-]$ such that the following version of the Jacobi identity is satisfied:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]],$$

for all $a, b, c \in V$.

Proposition 2.3.6. *Let A be a double Poisson algebra. Then*

$$\{a, \{\!\{b,c\}\!\}\} - \{\!\{\{a,b\},c\}\!\} - \{\!\{b,\{a,c\}\}\!\} = 0, \tag{2.3.7}$$

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}, \tag{2.3.8}$$

where, in (2.3.7), $\{a,-\}$ acts on tensors by $\{a, u \otimes v\} = \{a,u\} \otimes v + u \otimes \{a,v\}$, for all $a, u, v \in A$. In fact, $(A, \{-,-\})$ is a left Loday algebra.

Proof. All the statements are immediate consequences of [96], Proposition 2.4.2 (in fact, (2.3.8) is [96], Corollary 2.4.4). □

In [96], §2.7, M. Van den Bergh introduced the notion of *double Gerstenhaber algebra* (see, for instance, [96] Theorem 3.2.2). In the last part of this subsection, we shall define *double Poisson algebras of weight* $-N$ (the graded extension of double Poisson algebras) in such a way that Van den Bergh's double Gerstenhaber algebras will be double Poisson algebras of weight -1.

Let V_i, $i = 1, \ldots, n$ be graded vector spaces, $a = a_1 \otimes \cdots \otimes a_n$ a homogeneous element of $V_1 \otimes \cdots \otimes V_n$ and $s \in S_n$ be a permutation, then

$$\sigma_s(a) = (-1)^t a_{s^{-1}(1)} \otimes \cdots \otimes a_{s^{-1}(n)},$$

where

$$t = \sum_{\substack{i<j \\ s^{-1}(i)>s^{-1}(j)}} |a_{s^{-1}(i)}||a_{s^{-1}(j)}|.$$

Definition 2.3.9 (Double Poisson algebra of weight $-N$)**.** Let R be an associative k-algebra over a field of characteristic zero. Let A be a graded R-algebra and $N \in \mathbb{Z}_{\geq 0}$. A *double bracket of weight* $-N$ on A is a graded R-bilinear map

$$\{\!\!\{-,-\}\!\!\} : A \otimes A \longrightarrow A \otimes A$$

of weight $-N$ such that the following identities hold:

$$\{\!\!\{a, bc\}\!\!\} = (-1)^{(|a|-N)|b|} b \{\!\!\{a, c\}\!\!\} + \{\!\!\{a, b\}\!\!\} c, \tag{2.3.10}$$

$$\{\!\!\{a, b\}\!\!\} = -\sigma_{(12)}(-1)^{(|a|-N)(|b|-N)} \{\!\!\{b, a\}\!\!\}. \tag{2.3.11}$$

Furthermore, a double bracket of weight $-N$, $\{\!\!\{-,-\}\!\!\}$ on A, is a *double Poisson bracket of weight* $-N$ if satisfies the *graded double Jacobi identity*:

$$\begin{aligned} 0 = \{\!\!\{a, \{\!\!\{b, c\}\!\!\}\}\!\!\}_L + (-1)^{(|a|-N)(|b|+|c|)} \sigma_{(123)} \{\!\!\{b, \{\!\!\{c, a\}\!\!\}\}\!\!\}_L + \\ + (-1)^{(|c|-N)(|a|+|b|)} \sigma_{(132)} \{\!\!\{c, \{\!\!\{a, b\}\!\!\}\}\!\!\}_L . \end{aligned} \tag{2.3.12}$$

An algebra with a double Poisson bracket of weight $-N$ $(A, \{\!\!\{-,-\}\!\!\})$ is a *double Poisson algebra of weight* $-N$ (over R).

2.3.2 Poly-vector fields and the graded double Schouten–Nijenhuis bracket

A very useful description of Poisson brackets on a commutative smooth variety X is in terms of bivector fields, i.e., sections P of $\bigwedge^2 T_X$, such that $\{P, P\} = 0$, where the Schouten–Nijenhuis bracket $\{-,-\}$ determines a structure of Gerstenhaber algebra on the graded algebra $\bigwedge^\bullet T_X$ of poly-vector fields. Let R be an associative algebra and A be a graded smooth R-algebra, then we will now describe the non-commutative counterpart to the Schouten–Nijenhuis bracket on $\mathrm{T}_A \mathbb{D}\mathrm{er}_R A$.

Proposition 2.3.13. *Let homogeneous* $\Theta, \Delta \in \mathbb{D}\mathrm{er}_R A$*. Then*

$$\begin{aligned} \{\!\!\{\Theta, \Delta\}\!\!\}_l^{\sim} &= (\Theta \otimes 1)\Delta - (1 \otimes \Delta)\Theta, \\ \{\!\!\{\Theta, \Delta\}\!\!\}_r^{\sim} &= (1 \otimes \Theta)\Delta - (\Delta \otimes 1)\Theta = - \{\!\!\{\Delta, \Theta\}\!\!\}_l^{\sim}, \end{aligned}$$

define graded derivations $A \to A^{\otimes 3}$*, where the bimodule structure on* $A^{\otimes 3}$ *is the graded outer structure.*

Proof. It is an adaptation to the graded setting of [96], Proposition 3.2.1. □

Since A is a graded smooth R-algebra, by (iv) in §2.1.2, (2.2.30) and (2.2.34), we can define

$$\{\!\!\{\Theta, \Delta\}\!\!\}_l := \tau_{(23)} \circ \{\!\!\{\Theta, \Delta\}\!\!\}_l^{\sim}, \quad \{\!\!\{\Theta, \Delta\}\!\!\}_r := \tau_{(12)} \circ \{\!\!\{\Theta, \Delta\}\!\!\}_r^{\sim}$$

as elements of $\mathbb{D}\mathrm{er}_R A \otimes A$ and $A \otimes \mathbb{D}\mathrm{er}_R A$, respectively, and where $\tau_{(23)}$ and $\tau_{(12)}$ are given by (2.2.28) and (2.2.32), respectively.

Finally, given homogeneous $a, b \in A$ and $\Theta, \Delta \in \mathbb{D}\mathrm{er}_R A$, we define

$$\begin{aligned} \{\!\{a, b\}\!\} &= 0, \\ \{\!\{\Theta, a\}\!\} &= \Theta(a), \\ \{\!\{\Theta, \Delta\}\!\} &= \{\!\{\Theta, \Delta\}\!\}_l + \{\!\{\Theta, \Delta\}\!\}_r , \end{aligned} \tag{2.3.14}$$

with the right-hand sides in (2.3.14) viewed as elements of $(\mathrm{T}_A \mathbb{D}\mathrm{er}_R A)^{\otimes 2}$. The *graded double Schouten–Nijenhuis bracket* is the unique extension $\{\!\{-,-\}\!\} : (\mathrm{T}_A \mathbb{D}\mathrm{er}_R A)^{\otimes 2} \to (\mathrm{T}_A \mathbb{D}\mathrm{er}_R A)^{\otimes 2}$ of (2.3.14) of weight -1 to the tensor algebra $\mathrm{T}_A \mathbb{D}\mathrm{er}_R A$ satisfying the graded Leibniz rule (see (2.3.10)):

$$\{\!\{\Delta, \Theta\Phi\}\!\} = (-1)^{(|\Delta|-1)|\Theta|}\Theta \{\!\{\Delta, \Phi\}\!\} + \{\!\{\Delta, \Theta\}\!\} \Phi, \tag{2.3.15}$$

for homogeneous $\Delta, \Theta, \Phi \in \mathrm{T}_A \mathbb{D}\mathrm{er}_R A$.

2.3.3 Differential double Poisson algebras

Let B be an associative algebra and A be a graded smooth B-algebra. Then M. Van den Bergh [96, Proposition 4.1.2] proved that there exists an isomorphism between $(\mathrm{T}_A \mathbb{D}\mathrm{er}_B A)_n$ and B-linear n-brackets on A:

$$\mu \colon Q \longmapsto \{\!\{-, \cdots, -\}\!\}_Q , \tag{2.3.16}$$

which on $Q = \delta_1 \cdots \delta_n$ ($\delta_i \in \mathbb{D}\mathrm{er}_B A$ for all i) is given by

$$\{\!\{-, \cdots, -\}\!\}_Q = \sum_{i=0}^{n-1} (-1)^{(n-1)i} \tau^i_{(1\dots n)} \circ \{\!\{-, \cdots, -\}\!\}^{\sim}_Q \circ \tau^{-i}_{(1\dots n)}, \tag{2.3.17}$$

where

$$\{\!\{a_1, ..., a_n\}\!\}^{\sim}_Q = \delta_n(a_n)'\delta_1(a_1)'' \otimes \delta_1(a_1)'\delta_2(a_2)'' \otimes \cdots \otimes \delta_{n-1}(a_{n-1})'\delta_n(a_n)'').$$

It is important to note that the map μ in (2.3.16) factors through $\mathrm{T}_A \mathbb{D}\mathrm{er}_B A/[\mathrm{T}_A \mathbb{D}\mathrm{er}_B A, \mathrm{T}_A \mathbb{D}\mathrm{er}_B A]$. In addition, M. Van den Bergh proved in [96], Proposition 4.1.2 that if A is smooth over B, then μ is an isomorphism. Nevertheless, the calculation of $\{\!\{-, \cdots, -\}\!\}_Q$ using (2.3.17) may be difficult in practice. The following result overcomes this problem:

Proposition 2.3.18 ([96], Proposition 4.2.1)**.** *For $Q \in (\mathrm{T}_A \mathbb{D}\mathrm{er}_B A)_n$ (an n-fold of the tensor algebra), the following identity holds*

$$\{\!\{a_1, ..., a_n\}\!\}_Q = (-1)^{\frac{n(n-1)}{2}} \{\!\{a_1, ..., \{\!\{a_{n-1}, \{Q, a_n\}\}\!\}_L \cdots\}\!\}_L , \tag{2.3.19}$$

for all all $a_1, ..., a_n \in A$.

The subsequent result will be interesting because in the right hand side we shall obtain the double Jacobi identity for the bracket $\{\!\{-,-\}\!\}_P$:

Proposition 2.3.20 ([96], Proposition 4.2.2)**.** *Let $P \in (\mathrm{T}_A \mathbb{D}\mathrm{er}_B A)_2$. We have the following identity for $a, b, c \in A$:*

$$-\frac{1}{2} \{\!\{a, \{\!\{b, \{\{P, P\}, c\}\}\!\}\}\!\} =$$
$$= \{\!\{a, \{\!\{b, c\}\!\}_P\}\!\}_{P,L} + \tau_{(123)} \{\!\{b, \{\!\{c, a\}\!\}_P\}\!\}_{P,L} + \sigma_{(132)} \{\!\{c, \{\!\{a, b\}\!\}_P\}\!\}_{P,L} .$$

Hence this result enables us to introduce a new concept:

Definition 2.3.21 (DDP)**.** We say that A is a *differential double Poisson algebra* (a *DDP* for short) over B if it is equipped with an element $P \in (\mathrm{T}_A \mathbb{D}\mathrm{er}_B A)_2$ (a *differential double Poisson bracket*) such that

$$\{P, P\} = 0 \bmod [\mathrm{T}_A \mathbb{D}\mathrm{er}_B A, \mathrm{T}_A \mathbb{D}\mathrm{er}_B A]. \tag{2.3.22}$$

By Proposition 2.3.20, it is clear that if (A, P) is a differential double Poisson algebra then A is a double Poisson algebra with double bracket $\{\!\{-, -\}\!\}_P$. In the smooth case, since μ in (2.3.16) is an isomorphism, the notions of differential double Poisson algebra and double Poisson algebra coincide.

2.4 Bi-symplectic structures

Following [30] and [96], Appendix A, in this section we introduce the crucial notion of bi-symplectic algebras which we will extend to other settings below:

Definition 2.4.1. Let A be an associative k-algebra over a field of characteristic zero. An element $\omega \in \mathrm{DR}^2(A)$ is *bi-non-degenerate* if the map of A-bimodules (see (2.2.16))

$$\iota(\omega)\colon\ \mathbb{D}\mathrm{er}\, A \longrightarrow \Omega^1 A\colon \quad \Theta \longmapsto \iota_\Theta \omega$$

is an isomorphism. If in addition ω is closed in $\mathrm{DR}^\bullet(A)$, then we say that ω is *bi-symplectic*. An associative k-algebra A endowed with a bi-symplectic form ω (A, ω) will be called a *bi-symplectic algebra*.

Let $\omega \in \mathrm{DR}^2(A)$ be a bi-symplectic form. We define the *Hamiltonian double derivation* $H_a \in \mathbb{D}\mathrm{er}\, A$ corresponding to $a \in A$ via

$$\iota_{H_a} \omega = \mathrm{d}a, \tag{2.4.2}$$

and write

$$\{\!\{a, b\}\!\}_\omega = H_a(b) \in A \otimes A \tag{2.4.3}$$

Since $H_a(b) = i_{H_a}(\mathrm{d}b)$, (2.4.3) can be written in the following form:

$$\{\!\{a, b\}\!\}_\omega = i_{H_a} \iota_{H_b} \omega, \tag{2.4.4}$$

which is more convenient to prove that $\{\!\{-, -\}\!\}_\omega$ is a double bracket on A (see [96], Lemma A.3.2). Furthermore, the following result shows that double Poisson brackets on associative algebras and bi-symplectic algebras are closely related:

Lemma 2.4.5 ([96], Proposition A.3.3)**.** *If (A, ω) is a bi-symplectic algebra, then $\{\!\{-,-\}\!\}_\omega$ is a double Poisson bracket on A.*

Finally, the following result describes how the Hamiltonian double derivation interchanges double Poisson brackets and double Schouten–Nijenhuis brackets:

Proposition 2.4.6 ([96, Proposition 3.5.1])**.** *The following are equivalent:*

(i) $\{\!\{-,-\}\!\}$ *is a double Poisson bracket on A.*

(ii) $\{\!\{H_a, H_b\}\!\}_l = H_{\{\!\{a,b\}\!\}'} \otimes \{\!\{a,b\}\!\}''$, *for all $a, b \in A$.*

(iii) $\{\!\{H_a, H_b\}\!\}_r = \{\!\{a,b\}\!\}' \otimes H''_{\{\!\{a,b\}\!\}}$, *for all $a, b \in A$.*

(iv) $\{\!\{H_a, H_b\}\!\} = H_{\{\!\{a,b\}\!\}}$, *for all $a, b \in A$, where $H_x := H_{x'} \otimes x'' + x' \otimes H_{x''}$ for all $x = x' \otimes x'' \in A \otimes A$.*

2.5 Definition of bi-symplectic associative $\mathbb{N}$-algebras

In this subsection, let R be an associative k-algebra, with k a field of characteristic zero. Following [30], §2.7, the weights in A give rise to the *Euler derivation*

$$\mathrm{Eu}\colon A \longrightarrow A,$$

defined by $\mathrm{Eu}|_{A_j} = j \cdot \mathrm{Id}$ for $j = 0, 1, \ldots$. The action of the corresponding Lie derivative operator

$$\mathrm{L}_{\mathrm{Eu}}\colon \mathrm{DR}^\bullet_R(A) \to \mathrm{DR}^\bullet_R(A) \tag{2.5.1}$$

has nonnegative integral eigenvalues. As usual all canonical objects (forms, double derivations,...) acquire weights by means of this operator.

Definition 2.5.2. Let B be an associative R-algebra.

(i) An *associative* $\mathbb{N}$-*algebra* over B (shorthand for 'non-negatively graded algebra') is a $\mathbb{Z}$-graded associative B-algebra A such that $A^i = 0$ for all $i < 0$. We say $a \in A$ is *homogeneous of weight* $|a| = i$ if $a \in A^i$.

(ii) A *tensor* $\mathbb{N}$-*algebra* over B is an associative $\mathbb{N}$-algebra A over B which can be written as a tensor algebra $A = \mathrm{T}_B M$, for a positively graded B-bimodule M, so $M = \bigoplus_{i \in \mathbb{Z}} M^i$, where $M^i = 0$ for $i \leq 0$.

(iii) We say $m \in M$ is *homogeneous of weight* $|m| = i$ if $m \in M^i$.

(iv) The *weight* of an associative $\mathbb{N}$-algebra A is $|A| := \min_{S \in \mathcal{G}} \max_{a \in S} |a|$, where the elements of $\mathcal{G}$ are the finite sets of homogeneous generators of A.

Bi-symplectic forms over associative $\mathbb{N}$-algebras will be key ingredients in this thesis.

Definition 2.5.3. Let A be an associative $\mathbb{N}$-algebra over B. An element $\omega \in \mathrm{DR}^2_R(A)$ which is closed for the universal derivation d is a *bi-symplectic form of weight* N if

(i) $\mathrm{L_{Eu}}\omega = N\omega$ for the operator (2.5.1), and

(ii) the following map of graded A-bimodules is an isomorphism:

$$\iota(\omega)\colon \mathbb{D}\mathrm{er}_R A \longrightarrow \Omega^1_R A[-N]\colon \quad \Theta \longmapsto \iota_\Theta\omega.$$

An associative $\mathbb{N}$-algebra equipped with a bi-symplectic form of weight N (A,ω) is called a *bi-symplectic associative* $\mathbb{N}$*-algebra of weight* N (over B).

Following [98], [30] and §2.4, if $\omega \in \mathrm{DR}^2_R(A)$ is a bi-symplectic form of weight N, using the Hamiltonian double derivation $H_a \in \mathbb{D}\mathrm{er}_R A$ corresponding to an homogenous $a \in A$ (see (2.4.2)), we can write

$$\{\!\!\{a,b\}\!\!\}_\omega = i_{H_a}\iota_{H_b}\omega, \tag{2.5.4}$$

The crucial point for our construction is that $\{\!\!\{-,-\}\!\!\}_\omega$ just introduced is a double Poisson bracket of weight $-N$ (see Definition 2.3.9 and Proposition 2.4.5):

Lemma 2.5.5. *If* (A,ω) *is a bi-symplectic associative* $\mathbb{N}$*-algebra of weight* N *over* B*, then* $\{\!\!\{-,-\}\!\!\}_\omega$ *is a double Poisson bracket of weight* $-N$ *on* A*.*

Proof. It is a graded version of [96], Proposition A.3.3. To determine the weight of $\{\!\!\{-,-\}\!\!\}_\omega$, observe that by (2.4.2), $|H_a| + |\omega| = |a|$ and by (2.4.3), $|\{\!\!\{a,b\}\!\!\}_\omega| = |a| + |b| - |\omega|$. Thus, $\{\!\!\{-,-\}\!\!\}_\omega = -N$. □

Now, if ω is a bi-symplectic form of weight k on a graded R-algebra A, we say that an homogeneous double derivation $\Theta \in \mathbb{D}\mathrm{er}_R A$ is *bi-ymplectic* if the reduced Lie derivative (2.2.18) of ω along Θ vanishes. In other words, $\mathcal{L}_\Theta\omega = 0$. Moreover, as in the commutative case, bi-symplectic forms of weight k impose strong constrains over associative $\mathbb{N}$-algebras, as the following result shows:

Lemma 2.5.6. *Let* ω *be a bi-symplectic form of weight* $j \neq 0$ *on a associative* $\mathbb{N}$*-algebra* A *over* R*. Then*

(i) ω *is exact.*

(ii) *If* Θ *is a bi-symplectic double derivation of weight* l*, if* $j + l \neq 0$*, then* Θ *is a Hamiltonian double derivation.*

Proof. (i) Since ω is a bi-symplectic form of weight $j \neq 0$, by definition,

$$\mathrm{L_{Eu}}\omega = j\omega,$$

where $\mathrm{L_{Eu}}$ is the operator $\mathrm{L_{Eu}}\colon \mathrm{DR}^\bullet_R(A) \to \mathrm{DR}^\bullet_R(A)$ introduced in (2.5.1). Now, by the Cartan identity,

$$j\omega = \mathrm{L_{Eu}}\omega = \mathrm{d}i_{\mathrm{Eu}}\omega,$$

and (i) holds. To prove (ii), using that Θ is a bi-symplectic form and Lemma 2.2.19 we obtain

$$0 = \mathcal{L}_\Theta \omega = \mathrm{d}\iota_\Theta \omega. \tag{2.5.7}$$

We take $H := \mathrm{i}_{\mathrm{Eu}} \iota_\Theta \omega$, where $\mathrm{i}_{\mathrm{Eu}} \colon \mathrm{DR}^\bullet_R(A) \to \mathrm{DR}^\bullet_R(A)$ is the contraction operator because $\mathrm{Eu} \in \mathrm{Der}_R A$, we get

$$\mathrm{d}H = \mathrm{d}(\mathrm{i}_{\mathrm{Eu}} \iota_\Theta \omega) = \mathrm{L}_{\mathrm{Eu}}(\iota_\Theta \omega) = |\iota_\Theta \omega| \iota_\Theta \omega = (l+j)\iota_\Theta \omega,$$

where in the second identity we used (2.5.7). □

2.6 The Kontsevich–Rosenberg principle for bi-symplectic forms

This subsection is devoted to show how the Kontsevich–Rosenberg principle works for some geometric structures relevant for this thesis. We follow [30], §6.2, closely.

Let R be a semisimple finite-dimensional k-algebra over a field of characteristic zero k, and V an R-module. Observe that the action of R gives an algebra map $R \to \mathrm{Hom}_k(V,V)$, making $\mathrm{End}(V) := \mathrm{Hom}_k(V,V)$ an R-algebra. If A is a finitely generated associative R-algebra, $\mathrm{Rep}(A,V)$ is an affine scheme of finite type over k.

For sake of simplicity, we write $A_V := k[\mathrm{Rep}(A,V)]$. We consider $A_V \otimes \mathrm{End}(V)$, a tensor product of associative algebras, which also is an R-algebra via the map $R \to \mathrm{End}(V)$. Let $\mathrm{GL}(V)^R$ be the group of R-module automorphisms of V, and $\mathbb{G}_m \subset \mathrm{GL}(V)^R$ is the one-dimensional torus of scalar automorphisms of V. We define $G := \mathrm{GL}(V)^R/\mathbb{G}_m$. Observe that the action of G on V makes $\mathrm{Rep}(A,V)$ a G-scheme. This gives a G-action on A_V by algebra automorphisms. We also have a G-action on $\mathrm{End}(V)$, by conjugation, and G acts diagonally on $A_V \otimes \mathrm{End}(V)$. We write A_V^G and $(A_V \otimes \mathrm{End}(V))^G$ for the corresponding subalgebra of G-invariants.

To each element $a \in A$, we can associate the evaluation function $\widehat{a} \colon \mathrm{Rep}(A,V) \to \mathrm{End}(V) \colon \rho \mapsto \rho(a)$. The assignment $a \mapsto \widehat{a}$ gives rise an associative R-algebra homomorphism

$$\mathtt{ev} \colon A \to (A_V \otimes \mathrm{End}(V))^G \colon \quad a \longmapsto \widehat{a}.$$

If we compose the function $\widehat{a}$ with the trace map $\mathrm{Tr} \colon \mathrm{End}(V) \to k$ applied to the second tensor factor of $(A_V \otimes \mathrm{End}(V))^G$ above, we obtain a G-invariant element $\mathrm{Tr}\,\widehat{a} \in A_V^G$. If $a \in [A,A]$, then $\mathrm{Tr}\,\widehat{a} = 0$ because the symmetry of the trace. Therefore, the assignment $a \to \mathrm{Tr}\,\widehat{a}$ gives a well-defined k-linear map $\mathrm{Tr} \circ \mathtt{ev} \colon A/[A,A] \to A_V^G$. So, due to the Kontsevich–Rosenberg principle, we can regard $A/[A,A]$ as the space of non-commutative functions.

To study this principle for differential forms, we use a natural evaluation map on differential forms which sends the Karoubi de-Rahm complex of A to the ordinary de Rham complex of the representation scheme. We write

$$\Omega^\bullet(\mathrm{Rep}(A,V)) = \bigwedge\nolimits^{\bullet}_{A_V} \Omega^1(\mathrm{Rep}(A,V))$$

for the (ordinary) differential graded algebra of algebraic differential forms on the scheme $\mathrm{Rep}(A,V)$. We have an algebra homomorphism

$$\mathtt{ev}\colon \Omega^n A \to (\Omega^n(\mathrm{Rep}(A,V)) \otimes \mathrm{End}(V))^G : \alpha = a_0\, \mathrm{d}a_1 \ldots \mathrm{d}a_n \mapsto \widehat{\alpha} = \widehat{a}_0 \widehat{\mathrm{d}}\, a_1 \ldots \widehat{d}\, a_n,$$

defined as the following composite (observe that $\Omega^n A = A \otimes (A/k)^{\otimes n}$, see [31])

$$\begin{aligned} A \otimes (A/k)^{\otimes n} &\xrightarrow{\mathtt{ev}} (A_V \otimes \mathrm{End}(V)) \otimes \Omega^1(\mathrm{Rep}(A,V)) \otimes \mathrm{End}(V)^{\otimes n} \\ &\longrightarrow (\bigwedge\nolimits^{n}_{A_V} \Omega^1(\mathrm{Rep}(A,V))) \otimes (\mathrm{End}(V))^{\otimes n+1} \\ &\xrightarrow{\mathrm{Id}\otimes m} \Omega^n(\mathrm{Rep}(A,V)) \otimes \mathrm{End}(V). \end{aligned}$$

As $\widehat{r}$ is a constant function, for any $r \in R$, we have $\mathrm{d}\widehat{r} = 0$. It follows that the map above induces a well defined differential graded algebra morphism

$$\Omega^\bullet_R A \longrightarrow (\Omega^\bullet(\mathrm{Rep}(A,V)) \otimes \mathrm{End}(V))^G .$$

Furthermore, composing the latter morphism with the trace map $\mathrm{Id} \otimes \mathrm{Tr}\colon \Omega^\bullet(\mathrm{Rep}(A,V)) \otimes \mathrm{End}(V) \to \Omega^\bullet(\mathrm{Rep}(A,V))$, we obtain a linear map

$$\mathrm{Tr} \circ \mathtt{ev}\colon \mathrm{DR}^\bullet_R(A) \longrightarrow \Omega^\bullet(\mathrm{Rep}(A,V))^G : \quad \alpha \longmapsto \mathrm{Tr}\,\widehat{\alpha}.$$

Note that the previous map commutes with the de Rham differentials.

Next, we will apply the Kontsevich–Rosenberg principle to bi-symplectic forms. We will prove that $\mathrm{Tr}\,\widehat{\omega}$ is a symplectic form following closely [30], Theorem 6.4.3(ii).

We fix $\rho \in \mathrm{Rep}(A,V)$ and we see $V = V_\rho$ as an A-module and $\mathrm{End}\,V_\rho$ as an A-bimodule. Also, we denote $(\mathrm{End}\,V_\rho)^* := \mathrm{Hom}(V_\rho, k)$. Finally, observe that the trace pairing $(u,v) \mapsto \mathrm{Tr}(uv)$ gives an A-bimodule isomorphism $\mathrm{tr}\colon (\mathrm{End}\,V_\rho)^* \xrightarrow{\cong} \mathrm{End}\,V_\rho$.

Now, the Zariski tangent space to the affine scheme $\mathrm{Rep}(A,V)$ (see [22]) at the point $\rho \in \mathrm{Rep}(A,V)$ is

$$\mathrm{T}_\rho \mathrm{Rep}(A,V) = \mathrm{Der}_R(A, \mathrm{End}\,V_\rho) = \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, \mathrm{End}\,V_\rho).$$

Therefore, we can write the cotangent space at the same point $\rho \in \mathrm{Rep}(A,V)$:

$$\mathrm{T}^*_\rho \mathrm{Rep}(A,V) = (\mathrm{Der}_R(A, \mathrm{End}\,V_\rho))^* = (\mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, \mathrm{End}\,V_\rho))^*.$$

Let $\omega \in \mathrm{DR}^2_R(A)$. As we saw above, this 2-form gives rise to the reduced contraction map $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R A \to \Omega^1_R A$, and the corresponding 2-form $\mathrm{Tr}\,\widetilde{\omega} \in \Omega^2(\mathrm{Rep}(A,V)$, induces a similar contraction map $\widehat{i} = i(\mathrm{Tr}\,\widetilde{\omega})\colon \mathrm{T}_\rho \mathrm{Rep}(A,V) \to \mathrm{T}^*_\rho \mathrm{Rep}(A,V)$.

Assume now that A is smooth over R. Then we have the isomorphism (see §2.1.2)

$$\varphi_1\colon \mathbb{D}\mathrm{er}_R A \otimes_{A^{\mathrm{e}}} (\mathrm{End}\, V_\rho)^* \xrightarrow{\cong} \mathrm{Der}_R(A, (\mathrm{End}\, V_\rho)^*).$$

By the same reason, $\Omega^1_R A$ is a finitely generated projective A^{e}-module and it is easy to see that we have the following isomorphism

$$\varphi_2\colon \Omega^1_R A \otimes_{A^{\mathrm{e}}} (\mathrm{End}\, V_\rho)^* \xrightarrow{\cong} \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, \mathrm{End}\, V_\rho)^*.$$

Therefore, we can construct the following commutative diagram

$$\begin{tikzcd}
\mathbb{D}\mathrm{er}_R A \otimes_{A^{\mathrm{e}}} (\mathrm{End}\, V_\rho)^* \arrow[r, "\varphi_1"] \arrow[d, "\iota(\omega)\otimes \mathrm{Id}"] & \mathrm{Der}_R(A, (\mathrm{End}\, V_\rho)^*) \arrow[r, equal, "\mathrm{tr}"] & \mathrm{Der}_R(A, \mathrm{End}\, V_\rho) \arrow[r, equal] & \mathrm{T}^*_\rho \arrow[d, "\hat{i}"] \\
\Omega^1_R A \otimes_{A^{\mathrm{e}}} (\mathrm{End}\, V_\rho) \arrow[r, "\varphi_2"] & \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, \mathrm{End}\, V_\rho)^* \arrow[r, equal] & (\mathrm{Der}_R(A, \mathrm{End}\, V_\rho))^* \arrow[r, equal] & \mathrm{T}^*_\rho
\end{tikzcd}$$

where $\mathrm{T}_\rho := \mathrm{T}_\rho \mathrm{Rep}(A,V)$ and $\mathrm{T}^*_\rho := \mathrm{T}^*_\rho \mathrm{Rep}(A,V)$. Now, if ω is bi-non-degenerate, then the vertical map on the left of the diagram is a bijection. As a consequence, the vertical map on the right is a bijection as well. In this way, we can conclude that the 2-form $\mathrm{Tr}\,\widehat{\omega}$ is non-degenerate, as we required.

As final remarks, it is worthwhile to observe that instances of the Kontsevich–Rosenberg Principle are usually *ad hoc*, differing in each case. In order to reach a unified realization of this principle, M. Van den Bergh introduced in [97], Section 3.3, an additive functor $(-)_V\colon \mathrm{Mod}(A^{\mathrm{e}}) \to \mathrm{Mod}(A_V)$, which sends finitely generated A^{e}-modules to finitely generated A_V-modules. However, since $\mathrm{Rep}_V A$ are smooth schemes for all V if A is a formally smooth algebra, the Kontsevich–Rosenberg principle works well only when A is a formally smooth algebra.

To extend this principle to arbitrary algebras, there exists a very interesting program (see [56], [12], [14] and [13] and references therein) which proposes to replace $\mathrm{Rep}(A,V)$ by a differential graded scheme $\mathrm{DRep}(A,V)$ obtained by deriving the classical representation functor in the sense of Quillen's homotopical algebra. The idea (see, for instance, [13]) is that the transition from $\mathrm{Rep}(A,V)$ to $\mathrm{DRep}(A,V)$ means a desingularization of $\mathrm{Rep}(A,V)$, so it is expected that $\mathrm{DRep}(A,V)$ will play a role in the geometry of arbitrary non-commutative algebras similar to the role of $\mathrm{Rep}(A,V)$ in the geometry of smooth algebras.

Chapter 3

Restriction Theorems of Bi-symplectic forms

In this chapter we shall prove two technical results of graded bi-symplectic forms, roughly speaking corresponding to graded non-commutative versions, in weights 1 and 2, of the Darboux Theorem in symplectic geometry (as explained, for instance, in [18], §8.1). They turn out to be essential in subsequent chapters. In §3.1, following [31], we introduce the cotangent exact sequence relating absolute and relative differential forms, and also study its bidual (see Lemma 3.1.10). In §3.2, we introduce the crucial notion of bi-symplectic tensor $\mathbb{N}$-algebra of weight N (here $N \in \mathbb{N}^*$) which, in particular establishes an isomorphism between the space of double derivations and the bimodule of non-commutative differential 1-forms on the tensor $\mathbb{N}$-algebra given by ω, the bi-symplectic form of weight N. The first technical result is Theorem 3.2.2. It states that if (A, ω), with $A = \mathrm{T}_B M$, is a bi-symplectic tensor $\mathbb{N}$-algebra of weight N where R is a semisimple finite dimensional k-algebra, B is a smooth R-algebra and $M := E_1[-1] \oplus \cdots \oplus E_N[-N]$ for finitely generated projective B-bimodules E_i for all $1 \leq i \leq N$, the isomorphism $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R A \longrightarrow \Omega^1_R A[-N]$ restricts, in weight 0, to the B-bimodule isomorphism $\widetilde{\iota}(\omega)_{(0)}\colon \mathbb{D}\mathrm{er}_R B \xrightarrow{\cong} E_N$.

The second technical result is Theorem 3.3.40 where we carry out the construction of the isomorphism $\flat\colon E_1 \to E_1^\vee$ which turns out to be the restriction of the isomorphism $\iota(\omega)$ in weight 1. This Theorem is proved in the setting of graded double quivers (of weight 2) whose basics are reviewed in §3.3.1 (see Definition 3.3.4). In particular, since the graded path algebra of these objects can be expressed in terms of the graded tensor algebra of the bimodule V_P and as the graded tensor of the bimodule M_P (see (3.3.5) and (3.3.25)), in Lemma 3.3.7 we prove that there exists a canonical isomorphism between both descriptions. Finally, in Proposition 3.3.34, we show that graded double quivers are endowed with a canonical bi-symplectic form of even weight.

3.1 The cotangent exact sequence

From now on, we fix the following framework. Let R be a semi-simple associative k-algebra, where k is a field of characteristic zero, and B is an associative R-algebra. Consider $A = \mathrm{T}_B M$, where M is a graded B-bimodule and $\mathrm{T}_B(-)$ denotes the tensor algebra. Finally, $\omega \in \mathrm{DR}^2_R(A)_N$ will be a bi-symplectic form of weight N, where $N \in \mathbb{N} := \mathbb{Z}_{\geq 0}$.

Following [28] and [30], the tensor algebra $\mathrm{T}^* B := \mathrm{T}_B \mathbb{D}\mathrm{er}_R B$ of the B-bimodule $\mathbb{D}\mathrm{er}_R B$ is called the *noncommutative cotangent bundle* of B. Observe that it is a graded B-algebra, $\mathrm{T}^* B = \bigoplus_{i\geq 0} \mathrm{T}^*_i B$ such that $\mathrm{T}^*_0 B = B$, which can be regarded as the coordinate ring of the 'noncommutative cotangent bundle' on the 'noncommutative space' $\operatorname{Spec} B$ because its elements induce regular functions on the cotangent bundle $\mathrm{T}^* \operatorname{Rep}(B, V)$ of the representation scheme of B in any vector space V (see [30]).

The justification for this point of view is that for smooth B, $\mathrm{T}^* B$ carries a canonical Liouville 1-form $\lambda \in \mathrm{DR}^1_R(\mathrm{T}^* B)$ ([30], Proposition 5.2.4), which is a noncommutative analogue of the classical expression '$\lambda = p\, \mathrm{d}q$' (see [30] (5.2.7)), whose differential $\omega = \mathrm{d}\lambda \in \mathrm{DR}^2_R(\mathrm{T}^* B)$ is bi-symplectic ([30], Theorem 5.1.1) and induces the standard symplectic form on the cotangent bundles $\mathrm{T}^* \operatorname{Rep}(B, V)$ of the representation schemes.

Following [30] §5 and [31] §2, we shall describe a noncommutative analogue of the short exact sequence relating relative and absolute differential forms, sometimes called the first fundamental exact sequence (see, for instance, [72, Theorem 25.1]). We shall focus on a noncommutative analogue of this short exact sequence for (bi-symplectic) tensor $\mathbb{N}$-algebras.

In this subsection, R is a fixed associative k-algebra and consider B a smooth R-algebra. Throughout, an *R-algebra* is an associative k-algebra B equipped with a unit preserving k-algebra embedding $R \to B$. An *associative B-algebra* is then an R-algebra A equipped with an algebra homomorphism $B \to A$ compatible with the identity map $R \to R$ (in particular, it is unit preserving).

In this subsection, we fix that B is a graded associative R-algebra and M is a graded B-bimodule. We start by considering the relative differential forms of a tensor algebra

Proposition 3.1.1. *Let $A = \mathrm{T}_B M$. Then there exists a canonical isomorphism of graded A-bimodules $A \otimes_B M \otimes_B A \cong \Omega^1_B A \,:\, a_1 \otimes m \otimes a_2 \mapsto a_1(\mathrm{d}m)a_2$, where $\mathrm{d} : A \to \Omega^1_R A$ is the universal derivation.*

Proof. This result is a consequence of [31], Proposition 2.6 because the maps involved preserve weights. □

The cotangent exact sequence for an arbitrary graded associative B-algebra A is as follows.

Theorem 3.1.2. *There is a canonical exact sequence of graded A-bimodules*

$$0 \longrightarrow \mathrm{Tor}_1^B(A,A) \longrightarrow A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow \Omega^1_R A \longrightarrow \Omega^1_B A \longrightarrow 0.$$

Proof. This result is a consequence of [31], Proposition 2.6 because the maps involved preserve weights. □

Now, if we assume that M is a graded B-bimodule which is flat as either a left or right graded B-module and $A = \mathrm{T}_B M$, we simplify the previous exact short sequence using that $\mathrm{Tor}_1^B(A,A) = 0$ (and Proposition 3.1.1):

Corollary 3.1.3. *Let $A = \mathrm{T}_B M$, where M is a graded B-bimodule which is flat as either left or right graded B-module. Then we have an exact sequence of graded A-bimodules:*

$$0 \longrightarrow A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow \Omega^1_R A \longrightarrow A \otimes_B M \otimes_B A \longrightarrow 0$$

Proof. This result is a consequence of [31], Corollary 2.10 because the maps involved preserve weights. □

Now, with these ingredients, W. Crawley-Boevey, P. Etingof and V. Ginzburg provided a very explicit description of the space of noncommutative relative differential forms on A over R. Following [30] §5.2, we define the graded A-bimodule

$$\widetilde{\Omega} := (A \otimes_B \Omega^1_R B \otimes_B A) \bigoplus (A \otimes_R M \otimes_R A), \tag{3.1.4}$$

and abusing the notation, for any $a', a'' \in A$, $m \in M$, $\beta \in \Omega^1_R B$, we write

$$\begin{aligned} a' \cdot \widetilde{m} \cdot a'' &:= 0 \oplus (a' \otimes m \otimes a'') \in A \otimes_R M \otimes_R A \subset \widetilde{\Omega}, \\ a' \cdot \widetilde{\beta} \cdot a'' &:= (a' \otimes \beta \otimes a'') \oplus 0 \in A \otimes_B \Omega^1_R B \otimes_B A \subset \widetilde{\Omega}. \end{aligned}$$

Let $Q \subset \widetilde{\Omega}$ be the graded A-subbimodule generated by the Leibniz rule in $\widetilde{\Omega}$. In other words,

$$Q = \langle\!\langle \widetilde{b'mb''} - \widetilde{db'} \cdot (mb'') - b' \cdot \widetilde{m} \cdot b'' - (b'm) \cdot \widetilde{db''} \rangle\!\rangle_{b', b'' \in B, m \in M}, \tag{3.1.5}$$

where $\langle\!\langle - \rangle\!\rangle$ denotes the graded A-subbimodule generated by the set $(-)$. Clearly, the structure of graded algebra on $A = \mathrm{T}_B M$ induces a structure of graded A-bimodule on $\widetilde{\Omega}$ and $Q \subset \widetilde{\Omega}$ is a graded A-subbimodule, because it is generated by homogeneous elements, so the quotient $\widetilde{\Omega}/Q$ is a graded A-bimodule. We can now prove the following result which is a consequence of [30], Lemma 5.2.3 because weights are preserved:

Proposition 3.1.6. *Let B be a smooth graded R-algebra, M a finitely generated projective graded B-bimodule and $A = \mathrm{T}_B M$. Then*

(i) *There exists a graded A-bimodule isomorphism*

$$\Omega^1_R A \xrightarrow[f]{\simeq} \tilde{\Omega}/Q.$$

(ii) *The embedding of the first direct summand in $\widetilde{\Omega}$ (respectively the projection onto the second direct summand in $\widetilde{\Omega}$), induces, via the isomorphism in* (i), *a canonical extension of graded A-bimodules*

$$0 \longrightarrow A \otimes_B \Omega^1_R B \otimes_B A \xrightarrow{\varepsilon} \Omega^1_R A \xrightarrow{\nu} A \otimes_B M \otimes_B A \longrightarrow 0 \tag{3.1.7}$$

(iii) *The assignment $B \oplus M = \mathrm{T}^0_B M \oplus \mathrm{T}^1_B M \to \widetilde{\Omega}$, $b \oplus m \mapsto \widetilde{\mathrm{d}}b + \widetilde{m}$ extends uniquely to a graded derivation $\widetilde{\mathrm{d}}\colon A = \mathrm{T}_B M \to \widetilde{\Omega}/Q$; this graded derivation corresponds, via the isomorphism in* (i), *to the canonical universal graded derivation $\mathrm{d}\colon A \to \Omega^1_R A$. In other words, we have (see* (3.1.5)*)*

$$f(\widetilde{m}) = \mathrm{d}m, \quad f(\widetilde{\mathrm{d}b}) = \mathrm{d}b, \tag{3.1.8}$$

for homogeneous $m \in M$ and $b \in B$, and the commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\widetilde{\mathrm{d}}"'] \arrow[dr, "\mathrm{d}"] & \\ \tilde{\Omega}/Q \arrow[rr, "f", "\simeq"'] & & \Omega^1_R A \end{tikzcd}$$

Observe that under our hypothesis (B smooth and M a projective finitely generated B-bimodule), Proposition 3.1.7(i) states that there exists an isomorphism of graded A-bimodules $\Omega^1_R A \simeq \widetilde{\Omega}/Q$. Since $A = \mathrm{T}_B M$, we have to define this isomorphism (to be denoted by f) in $\mathrm{T}^0_B M = B$ and $\mathrm{T}^1_B M = M$ and, then, apply the universal property of tensor algebras (see, for example, [9], Lemma 1.2). For homogeneous $m \in M$ and $b \in B$, by (3.1.5),

$$f(\widetilde{m}) = \mathrm{d}m, \quad f(\widetilde{\mathrm{d}b}) = \mathrm{d}b, \tag{3.1.9}$$

where $\widetilde{\mathrm{d}}$: $A \longrightarrow \widetilde{\Omega}/Q$ is the graded derivation which corresponds via f to the canonical universal derivation d: $A \longrightarrow \Omega^1_R A$. In such a way, we have the commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\widetilde{\mathrm{d}}"'] \arrow[dr, "\mathrm{d}"] & \\ \tilde{\Omega}/Q \arrow[rr, "f", "\simeq"'] & & \Omega^1_R A \end{tikzcd}$$

Applying the functor $\mathrm{Hom}_{A^{\mathrm{e}}}(-{}_{A^{\mathrm{e}}}A^{\mathrm{e}})$ to (3.1.7), we obtain the “bidual cotangent sequence":

Lemma 3.1.10. *Let B be a smooth graded R-algebra, M a finitely generated projective graded B-bimodule and $A = \mathrm{T}_B M$. Then we have the following short exact sequence:*

$$0 \longrightarrow A \otimes_B M^\vee \otimes_B A \xrightarrow{\nu^\vee} \mathbb{D}\mathrm{er}_R A \xrightarrow{\varepsilon^\vee} A \otimes_B \mathbb{D}\mathrm{er}_R B \otimes_B A \longrightarrow 0 \tag{3.1.11}$$

Proof. Under our hypothesis, by Proposition 2.2.23, $A = \mathrm{T}_B\, M$ is graded smooth. Next, by the isomorphism `canonical` in (2.2.5), $\mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}})$ is isomorphic to $\mathbb{D}\mathrm{er}_R\, A$ as graded bimodules.

Now, using (2.1.27c), (2.1.28) and (2.1.29) which are isomorphisms under our hypothesis,

$$\begin{aligned}\mathrm{Hom}_{A^{\mathrm{e}}}(A\otimes_B \Omega^1_R B\otimes_B A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) &\simeq \mathrm{Hom}_{A^{\mathrm{e}}}(A^{\mathrm{e}}\otimes_{B^{\mathrm{e}}}\Omega^1_R B, {}_{A^{\mathrm{e}}}A^{\mathrm{e}})\\ &\simeq \mathrm{Hom}_{B^{\mathrm{e}}}(\Omega^1_R B, {}_{B^{\mathrm{e}}}A^{\mathrm{e}})\\ &\simeq \mathrm{Hom}_{B^{\mathrm{e}}}(\Omega^1_R B, B^{\mathrm{e}})\otimes_{B^{\mathrm{e}}} A^{\mathrm{e}}\\ &\simeq (\Omega^1_R B)^{\vee}\otimes_{B^{\mathrm{e}}} A^{\mathrm{e}}\\ &\simeq A\otimes_B \mathbb{D}\mathrm{er}_R\, B\otimes_B A.\end{aligned}$$

Similarly, we shall rely on the same kind of arguments in order to obtain the following description of $\mathrm{Hom}_{A^{\mathrm{e}}}(A\otimes_B M\otimes_B A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}})$:

$$\mathrm{Hom}_{A^{\mathrm{e}}}(A\otimes_B M\otimes_B A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) \simeq \mathrm{Hom}_{B^{\mathrm{e}}}(M, B^{\mathrm{e}})\otimes_{B^{\mathrm{e}}} A^{\mathrm{e}} \simeq A\otimes_B M^{\vee}\otimes_B A \qquad \square$$

Remark 3.1.12. To obtain the previous isomorphisms, we used the following isomorphism, where W is a $(B^{\mathrm{e}})^{\mathrm{op}}$-module:

$$\begin{aligned} h\colon W\otimes_{B^{\mathrm{e}}} A^{\mathrm{e}} &\xrightarrow{\cong} A\otimes_B W\otimes_B A\\ w\otimes(a_1\otimes a_2^{\mathrm{op}}) &\longmapsto (-1)^{|a_2|(|w|+|a_1|)} a_2\otimes g\otimes a_1,\end{aligned}$$

for $b_1\otimes b_2^{\mathrm{op}}\in B^{\mathrm{e}}$, $a_1\otimes a_2^{\mathrm{op}}\in A^{\mathrm{e}}$ and $w\in W$. In order to check that this map is a morphism of B^{e}-modules, we have to show that

$$h\left(w(b_1\otimes b_2^{\mathrm{op}})\otimes(a_1\otimes a_2^{\mathrm{op}}) - w\otimes(b_1\otimes b_2^{\mathrm{op}})(a_1\otimes a_2^{\mathrm{op}})\right)$$

is zero in $A\otimes_B W\otimes_B A$, which is straightforward using the well-known relations in this latter bimodule.

Proposition 3.1.13. *Let B a smooth graded algebra over R and M a projective finitely generated graded B-bimodule. Consider the tensor algebra $A = \mathrm{T}_B\, M$ endowed with a bi-symplectic form of weight N. Then the following diagram, where the rows are short exact sequences, commutes:*

(3.1.14)

$$\begin{tikzcd} 0 \arrow[r] & A\otimes_B M^{\vee}\otimes_B A \arrow[r, "\nu^{\vee}"] \arrow[d] & \mathbb{D}\mathrm{er}_R\, A \arrow[r, "\varepsilon^{\vee}"] \arrow[d, "\iota(\omega)"] & A\otimes_B \mathbb{D}\mathrm{er}_R\, B\otimes_B A \arrow[r] \arrow[d] & 0\\ 0 \arrow[r] & A\otimes_B \Omega^1_R B\otimes_B A \arrow[r, "\varepsilon"] & \Omega^1_R A \arrow[r, "\nu"] & A\otimes_B M\otimes_B A \arrow[r] & 0 \end{tikzcd}$$

Proof. It is an adaptation of [30], Lemma 5.4.2, when M is an arbitrary projective finitely generated graded B-bimodule. $\square$

3.2 Restriction Theorem in weight 0

In [86], Serre characterized algebraic vector bundles over an algebraic variety as finitely generated projective modules over the coordinate ring of the variety. Furthermore, Swan developed the topological counterpart of this result (see [92]): if X is a compact Hausdorff space, the category of complex vector bundles on X is equivalent to the category of finitely generated projective modules over $C(X)$, the algebra of continuous complex valued functions on X. The two theorems are collectively referred as the Serre–Swan theorem. So, if A is a not necessarily commutative algebra, we can think of a finitely generated projective A-module E as a *non-commutative vector bundle* over the non-commutative space represented by A. This point of view have turned out to be useful to develop a topological K-theory, a non-commutative Chern–Weil theory, and a Yang–Mills theory (see [57]).

If B is a (smooth) algebra, in this thesis, we will concern with B^{e}-modules because we shall consider tensor algebras $A = \mathrm{T}_B\, M$ of a finitely generated projective B^{e}-module M thanks of their nice algebraic properties as we showed in §3.1. Moreover, tensor algebras can be regarded as the non-commutative counterparts of symmetric algebras $\mathrm{Sym}^{\bullet}_B\, M$ (M an A^{e}-module) whose classification, when they are endowed with symplectic forms of weight 1 and 2, were carried out by Roytenberg in [81] (see §4.1). Hence the notion of bi-symplectic tensor $\mathbb{N}$-algebra of weight N will be particularly important in this thesis.

Definition 3.2.1. Let A be a tensor $\mathbb{N}$-algebra over B. An element $\omega \in \mathrm{DR}^2_R(A)$ of weight N which is closed for the universal derivation d is a *bi-symplectic form of weight* N if

(i) $\mathrm{L}_{\mathrm{Eu}}\omega = N\omega$ (see (2.5.1)),

(ii) the following map of graded A-bimodules is an isomorphism:

$$\iota(\omega)\colon\ \mathbb{D}\mathrm{er}_R\, A \longrightarrow \Omega^1_R(A)[-N]\colon \quad \Theta \longmapsto \iota_\Theta\omega.$$

A tensor $\mathbb{N}$-algebra equipped with a bi-symplectic form of weight N (A,ω) is called a *bi-symplectic tensor* $\mathbb{N}$ *algebra of weight* N over B if the underlying tensor $\mathbb{N}$-algebra can be written as $A = \mathrm{T}_B\, M$, where if $M = \bigoplus_{i\in\mathbb{N}} M^i$,

(a) $M^i = 0$ for $i > N$, and

(b) The underlying ungraded B-bimodule corresponding to M^i, with B the ungraded associative algebra, is finitely generated and projective, for $0 \leq i \leq N$.

In the following key result we shall prove that if (A,ω) is a bi-symplectic tensor algebra of weight N, the isomorphism $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R\, A \to \Omega^1_R(A)[-N]$ restricts to another one in weight zero:

Theorem 3.2.2. *Let R be a semisimple finite-dimensional k-algebra, B a smooth associative R-algebra and $E_1, \dots, E_N$ finitely generated projective B-bimodules, where $N > 0$. Define the tensor $\mathbb{N}$-algebra $A = \mathrm{T}_B M$ as the tensor B-algebra of the graded B-bimodule:*

$$M := M_1 \oplus \cdots \oplus M_N,$$

where $M_i := E_i[-i]$, for $i = 1, \dots, N$. Let $\omega \in \mathrm{DR}^2_R(A)$ be a bi-symplectic form of weight N over A. Then, the isomorphism $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R A \xrightarrow{\cong} \Omega^1_R A[-N]$ induces another isomorphism

$$\widetilde{\iota}(\omega)\colon A \otimes_B \mathbb{D}\mathrm{er}_R B \otimes_B A \xrightarrow{\cong} A \otimes_B M_N \otimes_B A,$$

which, in weight zero, restricts to the following isomorphism:

$$\widetilde{\iota}(\omega)_{(0)}\colon \mathbb{D}\mathrm{er}_R B \xrightarrow{\cong} E_N.$$

Remark 3.2.3. We shall now sketch the strategy used in the proof. Based on Proposition 3.1.13, we consider the following diagram:

$$\begin{tikzcd}
 & A \otimes_B M_N^\vee \otimes_B A \arrow[d, hook] & & & \\
0 \arrow[r] & A \otimes_B M^\vee \otimes_B A \arrow[r, "\nu^\vee"] & \mathbb{D}\mathrm{er}_R A \arrow[r, "\varepsilon^\vee"] \arrow[d, "\iota(\omega)"] & A \otimes_B \mathbb{D}\mathrm{er}_R B \otimes_B A \arrow[r] & 0 \\
0 \arrow[r] & A \otimes_B \Omega^1_R B \otimes_B A \arrow[r, "\varepsilon"] & \Omega^1_R A \arrow[r, "\nu"] & A \otimes_B M \otimes_B A \arrow[r] & 0
\end{tikzcd} \tag{3.2.4}$$

If we prove that the composition of maps

$$\begin{tikzcd}
A \otimes_B M_N^\vee \otimes_B A \arrow[d, hook] & & \\
A \otimes_B M^\vee \otimes_B A \arrow[r, "\nu^\vee"] & \mathbb{D}\mathrm{er}_R A \arrow[d, "\iota(\omega)"] & \\
 & \Omega^1_R A \arrow[r, "\nu"] & A \otimes_B M \otimes_B A
\end{tikzcd}$$

is zero then, by the universal property of the kernel, we obtain the arrow making the diagram (3.2.4) commutes:

$$\begin{tikzcd}
A \otimes_B M_N^\vee \otimes_B A \arrow[r, dashed] & A \otimes_B \Omega^1_R B \otimes_B A
\end{tikzcd}$$

Similarly, consider the "inverse diagram" obtained by means of the isomorphism $\iota(\omega)^{-1}$:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_B \Omega^1_R B \otimes_B A \arrow[r, "\varepsilon"] & \Omega^1_R A \arrow[r, "\nu"] \arrow[d, "\iota(\omega)^{-1}"] & A \otimes_B M \otimes_B A \arrow[r] & 0 \\
0 \arrow[r] & A \otimes_B M^\vee \otimes_B A \arrow[r, "\nu^\vee"] & \mathbb{D}\mathrm{er}_R A \arrow[r, "\varepsilon^\vee"] & A \otimes_B \mathbb{D}\mathrm{er}_R B \otimes_B A \arrow[r] & 0
\end{tikzcd}$$

If we show that the composition

$$\begin{tikzcd} A\otimes_B \Omega^1_R B\otimes_B A \arrow[r, "\varepsilon"] & \Omega^1_R A \arrow[d, "\iota(\omega)^{-1}"] & \\ & \mathbb{D}\mathrm{er}_R\, A \arrow[r, "\varepsilon^\vee"] & A\otimes_B \mathbb{D}\mathrm{er}_R\, B\otimes_B A \end{tikzcd}$$

is zero, using the same argument, we obtain that there exists a unique dashed arrow making the following diagram commutes,

$$\begin{tikzcd} 0 \arrow[r] & A\otimes_B \Omega^1_R B\otimes_B A \arrow[r, "\varepsilon"] \arrow[d, dashed] & \Omega^1_R A \arrow[r, "\nu"] \arrow[d, "\iota(\omega)^{-1}"] & A\otimes_B M\otimes_B A \arrow[r] & 0 \\ 0 \arrow[r] & A\otimes_B M^\vee\otimes_B A \arrow[r, "\nu^\vee"] & \mathbb{D}\mathrm{er}_R\, A \arrow[r, "\varepsilon^\vee"] & A\otimes_B \mathbb{D}\mathrm{er}_R\, B\otimes_B A \arrow[r] & 0 \end{tikzcd}$$

Moreover, from this morphism, we will be able to show that this dashed arrow restricts to the following one:

$$A\otimes_B \Omega^1_R B\otimes_B A \longrightarrow A\otimes_B M^\vee_N\otimes_B A$$

Putting all together:

$$\begin{tikzcd} A\otimes_B M^\vee_N\otimes_B A \arrow[r, dashed, shift left] & A\otimes_B \Omega^1_R B\otimes_B A \arrow[l, dashed, shift left] \end{tikzcd}$$

By construction, the dashed arrows are inverse to each other and so they are isomorphisms. Finally, the restriction to weight zero gives the result.

Proof. Consider the diagram (3.1.14) constructed in §3.1 and an element $a'\otimes\sigma\otimes a''\in A\otimes_B M^\vee_N\otimes_B A$, which from now on will be seen as an element of the space $A\otimes_B M^\vee\otimes_B A$ under the obvious injection $A\otimes_B M^\vee_N\otimes_B A\hookrightarrow A\otimes_B M^\vee\otimes_B A$, and where $a', a''\in A$ and $\sigma\in M^\vee_N$. Observe that its total weight is $|a'|+|a''|-N$ (in particular, $|\sigma|=-N$).

We shall write down explicitly the morphism $\nu^\vee$ of Proposition 3.1.13 by means of the isomorphisms κ and F which will be defined below. Also, these maps will make the square in the following diagram commutes:

(3.2.5)

$$\begin{tikzcd} & (A\otimes_B M\otimes_B A)^\vee \arrow[r, "\nu^\vee"] & (\widetilde{\Omega}/Q)^\vee \arrow[d, "F"] & & \\ 0 \arrow[r] & A\otimes_B M^\vee\otimes_B A \arrow[r] \arrow[u, "\kappa"] & \mathbb{D}\mathrm{er}_R\, A \arrow[r] & A\otimes_B \mathbb{D}\mathrm{er}_R\, B\otimes_B A \arrow[r] & 0 \end{tikzcd}$$

Firstly, define

$$\kappa\colon A\otimes_B M^\vee\otimes_B A \xrightarrow{\cong} (A\otimes_B M\otimes_B A)^\vee\colon \quad a'\otimes\sigma\otimes a''\longmapsto \kappa(a'\otimes\sigma\otimes a'')$$

given by

$$\begin{aligned} \kappa(a'\otimes\sigma\otimes a'')\colon A\otimes_B M\otimes_B A &\longrightarrow A\otimes A \\ a_1^{(1)}\otimes m_1\otimes a_1^{(2)} &\longmapsto \pm(a'\sigma''(m_1)a_1^{(2)})\otimes(a_1^{(1)}\sigma'(m_1)a''), \end{aligned}$$

where $a_1^{(1)}, a_1^{(1)} \in A$ and $m_1 \in M$. As usual, in this proof, we use Sweedler's convention. In addition, we will use the sign $\pm$ to indicate the signs involved since we will construct a map which turn out to be zero hence signs will be unnecessary for this purpose. Recall that by Proposition 3.1.6, the morphism ν is the natural projection:

$$\nu\colon \widetilde{\Omega}/Q \longrightarrow A \otimes_B M \otimes_B A\colon \begin{pmatrix} \overline{a}_2^{(1)} \otimes \beta_2 \otimes \overline{a}_2^{(2)} \\ + \\ a_2^{(1)} \otimes m_2 \otimes a_2^{(2)} \end{pmatrix} \operatorname{mod} Q \longmapsto a_2^{(1)} \otimes m_2 \otimes a_2^{(2)},$$

where $a_2^{(1)}, a_2^{(2)}, \overline{a}_2^{(1)}, \overline{a}_2^{(2)} \in A$, $m_2 \in M$ and $\beta_2 \in \Omega^1_R A$. Then define the map

$$\begin{aligned}\varphi &:= \kappa(a' \otimes \sigma \otimes a'') \left(\nu \begin{pmatrix} \overline{a}_2^{(1)} \otimes \beta_2 \otimes \overline{a}_2^{(2)} \\ + \\ a_2^{(1)} \otimes m_2 \otimes a_2^{(2)} \end{pmatrix} \operatorname{mod} Q \right) \in (\widetilde{\Omega}/Q)^\vee \\ &= \nu^\vee\left(\kappa(a' \otimes \sigma \otimes a'')\right)\end{aligned}$$

To define F in (3.2.5), we shall write $\widetilde{\mathrm{d}}$ acting on the homogeneous generators $b \in B = \mathrm{T}^0_B M$ and $m_i \in M = \mathrm{T}^1_B M$, for all $i = 1, ..., r$:

$$\widetilde{\mathrm{d}}a = \begin{cases} ((1_A \otimes \mathrm{d}_B b \otimes 1_A) \oplus 0) \operatorname{mod} Q & \text{if } a = b \\ (0 \oplus \sum_{i=1}^r m_1 \cdots m_{i-1} \otimes m_i \otimes m_{i+1} \cdots m_r) \operatorname{mod} Q & \text{if } a = m_1 \otimes \cdots \otimes m_r \end{cases}$$

These ingredients enable us to define

$$F\colon (\widetilde{\Omega}/Q)^\vee \longrightarrow \mathbb{D}\mathrm{er}_R A$$

given by

$$F(\varphi)(a) = \begin{cases} 0 & \text{if } a = b \\ (\varphi(\widetilde{\mathrm{d}}\, a))^\circ & \text{otherwise} \end{cases},$$

where $\varphi \in (\widetilde{\Omega}/Q)^\vee$ and $a \in A$. In particular, if $a = m_1 \otimes \cdots \otimes m_r$ with $r > 0$:

$$\begin{aligned}(\varphi(\widetilde{d}\, a))^\circ &= \sum_{i=1}^r \sigma_{(12)}\left((a'\sigma''(m_i)m_{i+1}\cdots m_r) \otimes (m_1 \cdots m_{i-1}\sigma'(m_i)a'')\right) \\ &= \sum_{i=1}^r (m_1 \cdots m_{i-1}\sigma'(m_i)a'') \otimes (a'\sigma''(m_i)m_{i+1}\cdots m_r)\end{aligned}$$

Claim 3.2.6. *$F(\varphi) \in \mathbb{D}\mathrm{er}_R A$.*

Proof. It is a straightforward application of the (graded) Leibniz rule. ☐

Next, we shall focus on the vertical arrow, $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R A \xrightarrow{\cong} \Omega^1_R A$ given by ω, the bi-symplectic form of weight N on A. Firstly, we use the canonical isomorphism $f^{-1}\colon \Omega^1_R A \simeq \widetilde{\Omega}/Q$ (see (3.1.9)), which induces another isomorphism $\Omega^2_R A \xrightarrow{\cong} (\widetilde{\Omega}/Q)^{\otimes_R 2}\colon \beta \longmapsto \widetilde{\beta}$. In particular, for the given bi-symplectic form ω, we obtain $\widetilde{\omega}$ which using $\widetilde{\Omega} = (A \otimes_B \Omega^1_R B \otimes_B A) \oplus (A \otimes_R M \otimes_R A)$ may decompose as follows:

$$\widetilde{\omega} = (\widetilde{\omega}_{\text{MM}} + \widetilde{\omega}_{\text{BB}} + \widetilde{\omega}_{\text{MB}} + \widetilde{\omega}_{\text{BM}}) \bmod Q,$$

and we write, dropping summation signs,

$$\begin{aligned}
\widetilde{\omega}_{\text{MM}} &= (\widetilde{m}_1 \otimes \widetilde{m}_2) \bmod Q,\\
\widetilde{\omega}_{\text{BB}} &= (\widetilde{\beta}_1 \otimes \widetilde{\beta}_2) \bmod Q,\\
\widetilde{\omega}_{\text{MB}} &= (\widetilde{m}_3 \otimes \widetilde{\beta}_3) \bmod Q,\\
\widetilde{\omega}_{\text{BM}} &= (\widetilde{\beta}_4 \otimes \widetilde{m}_4) \bmod Q.
\end{aligned}$$

where $\widetilde{m}_i := a_i^{(1)} \otimes m_i \otimes a_i^{(2)} \in A \otimes_R M \otimes_R A$ and $\widetilde{\beta}_i := \overline{a}_i^{(1)} \otimes \beta_i \otimes \overline{a}_i^{(2)} \in A \otimes_B \Omega^1_R B \otimes_B A$ for $i = 1, 2, 3, 4$, with $a_i^{(1)}, a_i^{(2)}, \overline{a}_i^{(1)}, \overline{a}_i^{(2)} \in A$, $m_i \in M$ and $\beta_i \in \Omega^1_R B$ for $i = 1, 2, 3, 4$. Keeping in mind this decomposition and the previous isomorphism, we can calculate:

$$\iota_{F(\varphi)}\widetilde{\omega} = \iota(\widetilde{\omega})(F(\varphi)) = \iota\left((\widetilde{\omega}_{\text{MM}} + \tilde{\omega}_{\text{BB}} + \tilde{\omega}_{\text{MB}} + \tilde{\omega}_{\text{BM}}) \bmod Q\right)(F(\varphi)) \tag{3.2.7}$$

Claim 3.2.8. *With the previous notation,*

(i) $\iota_{F(\varphi)}(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) = \begin{cases} 0 & \textit{if } |m_i| < N \\ a_i^{(1)\circ}(a'\sigma''(m_i) \otimes \sigma'(m_i)a'')a_i^{(2)} & \textit{if } |m_i| = N \end{cases}.$

(ii) $\iota_{F(\varphi)}(\bar{a}_i^{(1)} \otimes \beta \otimes \bar{a}_i^{(2)}) = 0$

Proof. Observe that we do not know how the operator $\iota_{F(\varphi)}$ acts on elements of $\widetilde{\Omega}/Q$ but we do how it acts on elements of $\Omega^1_R A$. Keeping this in mind, we have to use the canonical isomorphism f between these objects and then apply the operator $\iota_{F(\varphi)}$.

(i) Firstly,

$$\begin{aligned}
\iota_{F(\varphi)}(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) &= \iota_{F(\varphi)}(a_i^{(1)}(\mathrm{d}_A m_i)a_i^{(2)})\\
&= a_i^{(1)}\iota_{F(\varphi)}(\mathrm{d}_A m_i)a_i^{(2)}\\
&= a_i^{(1)\circ}(F(\varphi)(m_i))a_i^{(2)}
\end{aligned}$$

Hence, we have to distinguish two cases:

(a) **Case** $|m_i| < N$**:** Since $\sigma \in M_N^\vee$, $\sigma(m_i) = 0$ since $(A \otimes A)_{(j)} = \{0\}$ with $j < 0$. Thus, $\iota_{F(\varphi)}(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) = 0$.

(b) **Case** $|m_i| = N$**:**

$$\begin{aligned}
\iota_{F(\varphi)}(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) &= a_i^{(1)\circ}(F(\varphi)(m_i))a_i^{(2)}\\
&= a_i^{(1)\circ}(a'\sigma''(m_i) \otimes \sigma'(m_i)a'')a_i^{(2)} \in A
\end{aligned}$$

(ii) This case is quite similar; by definition of $F(\varphi)$:

$$\begin{aligned}
\iota_{F(\varphi)}(\overline{a}_i^{(1)} \otimes \beta_i \otimes \overline{a}_i^{(2)}) &= \iota_{F(\varphi)}(\overline{a}_i^{(1)}(b_i^{(1)}\, \mathrm{d}_A b_i^{(2)})\overline{a}^{(2)}) \\
&= \bar{a}_i^{(1)} b_i^{(1)\circ}(F(\varphi)(b_i^{(2)}))\overline{a}_i^{(2)} = 0
\end{aligned}$$

□

Now, we shall use the Claim 3.2.8 to analyze each summand in (3.2.7):

- **Case $\widetilde{\omega}_{\mathbf{BB}}$:**

  As $|\widetilde{\omega}_{\mathrm{BB}}| = N$ and $|\beta_1| = |\beta_2| = 0$, $|\overline{a}_i^{(1)}| + |\overline{a}_i^{(2)}| = N$, with $\overline{a}_i^{(1)}, \overline{a}_i^{(2)} \in A$ for $i = 1, 2$. Without loss of generality, in this case, we can assume that $|\overline{a}_1^{(1)}| = N$. Then,

$$\begin{aligned}
\iota(\widetilde{\omega}_{\mathrm{BB}})(F(\varphi)) &= \iota_{F(\varphi)}(\widetilde{\omega}_{\mathrm{BB}}) \\
&= \iota_{F(\varphi)}(\widetilde{\beta}_1 \otimes \widetilde{\beta}_2) \\
&= \left(\iota_{F(\varphi)}(\widetilde{\beta}_1)\right) \widetilde{\beta}_2 + \widetilde{\beta}_1 \left(\iota_{F(\varphi)}(\widetilde{\beta}_2)\right) \\
&= \left(\iota_{F(\varphi)}(\overline{a}_1^{(1)} \otimes \beta_1 \otimes \overline{a}_1^{(2)})\right) \widetilde{\beta}_2 + \widetilde{\beta}_1 \left(\iota_{F(\varphi)}(\overline{a}_2^{(1)} \otimes \beta_2 \otimes \overline{a}_2^{(2)})\right) \\
&= 0
\end{aligned}$$

- **Case $\widetilde{\omega}_{\mathbf{MM}}$:**

  As $|\widetilde{\omega}_{\mathrm{MM}}| = N$ and $|m_i| \geq 1$, then $|m_i| < N$ for $i = 1, 2$. Then, using Claim 3.2.8:

$$\begin{aligned}
\iota(\widetilde{\omega}_{\mathrm{MM}})(F(\varphi)) &= \iota_{F(\varphi)}(\widetilde{\omega}_{\mathrm{MM}}) \\
&= \iota_{F(\varphi)}(\widetilde{m}_1 \otimes \widetilde{m}_2) \\
&= \left(\iota_{F(\varphi)}(\widetilde{m}_1)\right) \widetilde{m}_2 + \widetilde{m}_1 \left(\iota_{F(\varphi)}(\widetilde{m}_2)\right) \\
&= \left(\iota_{F(\varphi)}(a_1^{(1)} \otimes m_1 \otimes a_1^{(2)})\right) \widetilde{m}_2 + \widetilde{m}_1 \left(\iota_{F(\varphi)}(a_2^{(1)} \otimes m_2 \otimes a_2^{(2)})\right) \\
&= 0.
\end{aligned}$$

- **Case $\widetilde{\omega}_{\mathbf{MB}}$:**

  In this case, $|\beta_3| = 0$, so $N \geq |m_3| \geq 1$. Again, by the Leibniz rule and Claim 3.2.8:

$$\begin{aligned}
\iota(\widetilde{\omega}_{\mathrm{MB}})(F(\varphi)) &= \iota_{F(\varphi)}(\widetilde{\omega}_{\mathrm{MB}}) \\
&= \iota_{F(\varphi)}(\widetilde{m}_3 \otimes \widetilde{\beta}_3) \\
&= \left(\iota_{F(\varphi)}(\widetilde{m}_3)\right) \widetilde{\beta}_3 + \widetilde{m}_3 \left(\iota_{\delta(\varphi)}(\widetilde{\beta}_3)\right) \\
&= \left(\iota_{F(\varphi)}(a_3^{(1)} \otimes m_3 \otimes a_3^{(2)})\right) \widetilde{\beta}_3 + \widetilde{m}_3 \left(\iota_{F(\varphi)}(\overline{a}_3^{(1)} \otimes \beta_3 \otimes \overline{a}_3^{(2)})\right) \\
&= \left(a_3^{(1)\circ}(F(\varphi)(m_3))a_3^{(2)}\right) \widetilde{\beta}_3
\end{aligned}$$

  Now we have to distinguish two cases depending on the weight of m_3:

(a) **Case** $|m_3| < N$**:** by Claim 3.2.8,

$$\iota(\widetilde{\omega}_{\mathrm{MB}})(F(\varphi)) = 0.$$

(b) **Case** $|m_3| = N$**:** by the same Claim,

$$\begin{aligned}\iota(\widetilde{\omega}_{\mathrm{MB}})(F(\varphi)) &= \left(a_3^{(1)\circ}(a'\sigma''(m_3)\otimes\sigma'(m_3)a'')a_3^{(2)}\right)\widetilde{\beta}_3\\ &\in \big((A\otimes_B \Omega^1_R B\otimes_B A)\oplus 0\big) \bmod Q \subset \widetilde{\Omega}/Q\end{aligned}$$

- **Case** $\widetilde{\omega}_{\mathbf{MB}}$

  It is quite similar to the previous case.

So, in conclusion, $\iota(\widetilde{\omega})(F(\varphi)) \in (A\otimes_B \Omega^1_R B\otimes_B A\oplus 0) \bmod Q$. The last step consists of defining the map g making the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A\otimes_B \Omega^1_R B\otimes_B A \arrow[r, "\varepsilon"] & \Omega^1_R A \arrow[r, "\nu"] \arrow[d, "\simeq"] & A\otimes_B M\otimes_B A \arrow[r] & 0\\
 & & \widetilde{\Omega}/Q \arrow[ur, "g"] & &
\end{tikzcd}$$

By Proposition 3.1.6, we know that ν is the projection onto the second direct summand of $\widetilde{\Omega}/Q$ so $g(\iota(\widetilde{\omega}_{\mathrm{MB}})(F(\varphi)))$ is zero in $A\otimes_B M\otimes_B A$. The universal property of the kernel allows us to conclude the existence of the dashed maps

$$\begin{tikzcd}
 & A\otimes_B M_N^\vee\otimes_B A \arrow[d, hook] & & & \\
0 \arrow[r] & A\otimes_B M^\vee\otimes_B A \arrow[r, "\nu^\vee"] \arrow[d, dashed] & \mathbb{D}\mathrm{er}_R A \arrow[r, "\varepsilon^\vee"] \arrow[d, "\iota(\omega)"] & A\otimes_B \mathbb{D}\mathrm{er}_R B\otimes_B A \arrow[r] \arrow[d, dashed] & 0\\
0 \arrow[r] & A\otimes_B \Omega^1_R B\otimes_B A \arrow[r, "\varepsilon"] & \Omega^1_R A \arrow[r, "\nu"] & A\otimes_B M\otimes_B A \arrow[r] & 0
\end{tikzcd}$$

making this diagram commutes. Finally, it follows that we constructed the following map:

$$A\otimes_B M_N^\vee\otimes_B A \dashrightarrow A\otimes_B \Omega^1_R B\otimes_B A \tag{3.2.9}$$

Next, we shall consider the 'inverse' diagram:

$$\begin{tikzcd}
0 \arrow[r] & A\otimes_B \Omega^1_R B\otimes_B A \arrow[r, "\varepsilon"] & \Omega^1_R A \arrow[r, "\nu"] \arrow[d, "\iota(\omega)^{-1}"] & A\otimes_B M\otimes_B A \arrow[r] & 0\\
0 \arrow[r] & A\otimes_B M^\vee\otimes_B A \arrow[r, "\nu^\vee"] & \mathbb{D}\mathrm{er}_R A \arrow[r, "\varepsilon^\vee"] & A\otimes_B \mathbb{D}\mathrm{er}_R B\otimes_B A \arrow[r] & 0
\end{tikzcd} \tag{3.2.10}$$

In a first stage, our aim is to construct the following dashed arrow:

$$A\otimes_B \Omega^1_R B\otimes_B A \dashrightarrow A\otimes_B M^\vee\otimes_B A$$

which makes the previous diagram commutative.

We begin by recalling that since $\widetilde{\Omega} = (A \otimes_B \Omega^1_R B \otimes_B A) \oplus (A \otimes_R M \otimes_R A)$, h is the imbedding of the first direct summand in $\widetilde{\Omega}$ (see Proposition 3.1.6), `proj` is the natural projection and the isomorphism f was defined in (3.1.9),

$$\begin{tikzcd}
 & & & \tilde{\Omega} \arrow[d, two heads, "\mathtt{proj}"] & & \\
 & & & \tilde{\Omega}/Q \arrow[d, "f"] & & \\
0 \arrow[r] & A \otimes_B \Omega^1_R B \otimes_B A \arrow[uurr, "h"] \arrow[rr, "\varepsilon"] & & \Omega^1_R A \arrow[r, "\nu"] & A \otimes_B M \otimes_B A \arrow[r] & 0
\end{tikzcd}$$

Let $a', a'' \in A$, $b \in B$ and $\mathrm{d}_B b \in \Omega^1_R B$. Then $a' \otimes \mathrm{d}_B b \otimes a'' \in A \otimes_B \Omega^1_R B \otimes_B A$. It is a simple calculation that

$$\varepsilon \colon A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow \Omega^1_R A \colon \quad a' \otimes \mathrm{d}_B b \otimes a'' \longmapsto a'(\mathrm{d}_A b) a'' \tag{3.2.11}$$

Now, we focus on the vertical arrow of the diagram (3.2.10). Observe that since ω is a bi-symplectic form of weight N, $\iota(\omega)^{-1}$ has weight $-N$. In fact, using (2.4.3), we can write this double Poisson bracket in terms of the Hamiltonian double derivation. Nevertheless, since $\{\!\!\{-,-\}\!\!\}_\omega$ is A-bilinear with respect to the outer bimodule structure on $A \otimes A$ in the second argument and A-bilinear with respect to the inner bimodule structure on $A \otimes A$ in the first argument, it is enough to consider $a' = a'' = 1_A$. Then

$$\left(\iota(\omega)^{-1} \circ \varepsilon\right)(1_A \otimes \mathrm{d}_B b \otimes 1_A) = \{\!\!\{b, -\}\!\!\}_\omega = H_b \tag{3.2.12}$$

Observe that $H_b \in \mathbb{D}\mathrm{er}_R A$ has weight $-N$. Finally, since `inj` is the imbedding of $A \otimes_B \Omega^1_R B \otimes_B A$ in the first direct summand of $\widetilde{\Omega}$, we shall determine Ψ to the square in the following diagram commutes: :

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_B M^\vee \otimes_B A \arrow[r, "\varepsilon^\vee"] & \mathbb{D}\mathrm{er}_R A \arrow[d, "\Psi"] \arrow[r, "\nu^\vee"] & A \otimes_B \mathbb{D}\mathrm{er}_R B \otimes_B A \arrow[r] & 0 \\
 & & (\Omega^1_R A)^\vee \arrow[d, "f^\vee"] & & \\
 & & (\widetilde{\Omega}/Q)^\vee \arrow[d, two heads, "\mathtt{proj}^\vee"] & & \\
 & & (\widetilde{\Omega})^\vee \arrow[r, hook, "(\mathtt{inj})^\vee"] & (A \otimes_B \Omega^1_R A \otimes_B A)^\vee \arrow[uuu, "\simeq"'] &
\end{tikzcd}$$

In this diagram, we define

$$\Psi \colon \mathbb{D}\mathrm{er}_R A \longrightarrow (\Omega^1_R A)^\vee \colon \quad \Theta \longmapsto \Psi(\Theta)$$

given by

$$\Psi(\Theta)\colon \Omega^1_R A \longrightarrow A\otimes A\colon \quad \alpha \longmapsto \Psi(\Theta)(\alpha) = (i_\Delta \alpha)^\circ = \pm i''_\Theta(\alpha)\otimes i'_\Theta(\alpha),$$

where $\pm := (-1)^{(\|i'_\Theta(\alpha)\|\|i''_\Theta(\alpha)\|+|i'_\Theta(\alpha)||i''_\Theta(\alpha)|)}$ (see (2.2.14)). When we apply Ψ to the element in (3.2.12):

$$\Psi\colon \mathbb{D}\mathrm{er}_R A \longrightarrow (\Omega^1_R A)^\vee\colon \quad H_b \longmapsto i_{H_b}, \tag{3.2.13}$$

such that

$$\begin{aligned} i_{H_b}\colon \Omega^1_R A &\longrightarrow A\otimes A \\ \bar{c}_1\,\mathrm{d}_A\bar{c}_2 &\longmapsto (\bar{c}_1 H_b(\bar{c}_2))^\circ \end{aligned} \tag{3.2.14}$$

Next, applying $f^\vee$, we obtain the following:

$$(f^\vee\circ\Psi)(H_b)\colon \widetilde{\Omega}/Q \to A\otimes A$$
$$\begin{pmatrix} \bar{a}_2^{(1)}\otimes b_1^{(1)}\,\mathrm{d}_B b_1^{(2)}\otimes \bar{a}_2^{(2)} \\ + \\ a_2^{(1)}\otimes m\otimes a_2^{(2)} \end{pmatrix} \bmod Q \mapsto \begin{pmatrix} \left(\bar{a}_2^{(1)} b_1^{(1)} H_b(b_1^{(2)})\bar{a}_2^{(2)}\right)^\circ \\ + \\ \left(a_2^{(1)} H_b(m) a_2^{(2)}\right)^\circ \end{pmatrix} \tag{3.2.15}$$

To shorten the notation, we make the following definition:

$$L := \left(\mathtt{proj}^\vee\circ f^\vee\circ\Psi\right)(H_b).$$

Finally, since

$$\begin{aligned} \mathtt{inj}^\vee\circ L\colon A\otimes_B \Omega^1_R A\otimes_B A &\longrightarrow A\otimes A \\ \bar{a}_2^{(1)}\otimes b_1^{(1)}\,\mathrm{d}_B b_1^{(2)}\otimes \bar{a}_2^{(2)} &\longmapsto \left(\bar{a}_2^{(1)} b_1^{(1)} H_b(b_1^{(2)})\bar{a}_2^{(2)}\right)^\circ \end{aligned}$$

The key point is to observe that since $b, b_1^{(2)}\in B$, $|b| = |b_1^{(2)}| = 0$. Thus, $|H_{b''}(b_1^{(2)})| = -N < 0$. Thus, $H_b(b_1^{(2)}) = 0$ because A is a bi-symplectic tensor $\mathbb{N}$-algebra so, in particular, it is non-negatively graded. By the universal property of the kernel, we conclude the existence of the dashed arrows which makes the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A\otimes_B \Omega^1_R B\otimes_B A \arrow[r, "\varepsilon"] \arrow[d, dashed] & \Omega^1_R A \arrow[r, "\nu"] \arrow[d, "\iota(\omega)^{-1}"] & A\otimes_B M\otimes_B A \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & A\otimes_B M^\vee\otimes_B A \arrow[r, "\nu^\vee"] & \mathbb{D}\mathrm{er}_R A \arrow[r, "\varepsilon^\vee"] & A\otimes_B \mathbb{D}\mathrm{er}_R B\otimes_B A \arrow[r] & 0 \end{tikzcd}$$

In special, we are interested in the map

$$A\otimes_B \Omega^1_R B\otimes_B A \dashrightarrow A\otimes_B M^\vee\otimes_B A \tag{3.2.16}$$

Finally, in (3.2.15), we point out that $\left(a_2^{(1)} H_b(m) a_2^{(2)}\right)^\circ = 0$ unless $m\in M_N$ since $|H_b| = |m| - N$. Hence, as a consequence of this discussion and using (3.2.16), we obtain:

$$A\otimes_B \Omega^1_R B\otimes_B A \dashrightarrow A\otimes_B M_N^\vee\otimes_B A \tag{3.2.17}$$

By construction, (3.2.9) and (3.2.17) are inverse to each other. So, we proved the existence of the following isomorphism:

$$A \otimes_B \Omega^1_R B \otimes_B A \simeq A \otimes_B M_N^{\vee} \otimes_B A$$

Or, equivalently using the fact that, by hypothesis, B is a smooth associative R-algebra,

$$A \otimes_B \mathbb{D}\mathrm{er}_R\, B \otimes_B A \simeq A \otimes_B M_N \otimes_B A \tag{3.2.18}$$

For reasons that we will clarify below, we make precise this isomorphism:

Claim 3.2.19. *Let (A, ω) be a bi-symplectic tensor $\mathbb{N}$-algebra of weight N. Then $\iota(\omega)^{-1}$ restricts to a B-bimodule isomorphism*

$$E_N \simeq \mathbb{D}\mathrm{er}_R\, B. \tag{3.2.20}$$

Proof. Observe that in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_B \Omega^1_R B \otimes_B A \arrow[r, "\varepsilon"] \arrow[d, dashed] & \Omega^1_R A \arrow[r, "\nu"] \arrow[d, "\iota(\omega)^{-1}"] & A \otimes_B M \otimes_B A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes_A M^{\vee} \otimes_B A \arrow[r, "\nu^{\vee}"] & \mathbb{D}\mathrm{er}_R\, A \arrow[r, "\varepsilon^{\vee}"] & A \otimes_B \mathbb{D}\mathrm{er}_R\, B \otimes_B A \arrow[r] & 0
\end{tikzcd}$$

the dashed arrow is $\iota(\omega)^{-1} \circ \varepsilon$. We shall see that the weight of this map is -N. This is equivalent to prove that $|\varepsilon| = 0$ since ω is a bi-symplectic form of weight N. As we discussed in (3.2.11),

$$\varepsilon \,:\, A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow \Omega^1_R A\colon \quad a' \otimes b'\, \mathrm{d}_B b'' \otimes a'' \longmapsto a'(b'\, \mathrm{d}_A b'')a''.$$

Thus, it is immediate that $|\varepsilon| = 0$.

Finally, observe that A is non-negatively graded and M_N has weight N while $\mathbb{D}\mathrm{er}_R\, B$ has weight 0. Note that the part of weight 0 of $A \otimes_B \mathbb{D}\mathrm{er}_R\, B \otimes_B A$ is $B \otimes_B \mathbb{D}\mathrm{er}_R\, B \otimes_B B$ which is isomorphic to $\mathbb{D}\mathrm{er}_R\, B$. Similarly, $(A \otimes_B M_N \otimes_B A)_N = B \otimes_B M_N \otimes_B B$, where $(-)_N$ denotes the part of weight N. Thus, we obtain the following isomorphism of B-bimodules,

$$E_N \simeq \mathbb{D}\mathrm{er}_R\, B \qquad \square$$

Claim 3.2.19 finishes the proof of the Theorem 3.2.2. $\square$

3.3 Restriction Theorem in weight 1

Let k be a field of characteristic zero, R an associative k-algebra and B a smooth algebra. Let (A, ω) be a bi-symplectic tensor algebra over B of weight 2. This section is devoted to prove Theorem 3.3.40, where we shall prove that the isomorphism $\iota(\omega)\colon \mathbb{D}\mathrm{er}_R A \to \Omega^1_R A[-2]$ restricts, in weight 1, to the isomorphism of B-bimodules $(\iota(\omega))_1\colon E_1 \xrightarrow{\cong} E_1^\vee$ (here E_1 is a finitely generated projective B-bimodule). We will prove this result using double graded quivers, whose basics are developed in §3.3.1–§3.3.4. In particular, we will show that a double graded quiver of even weight is endowed with a canonical bi-symplectic form (see Proposition 3.3.34).

3.3.1 Background on graded quivers

Quivers

In this subsection, we establish some well-known notions and results which enable us to fix notation. We will closely follow the modern references [7] and [9].

A *quiver* Q consists of a set Q_0 of vertices, a set Q_1 of arrows and two maps $t, h\colon Q_1 \to Q_0$ assigning to each arrow $a \in Q_1$, its *tail* and its *head*. We write $a\colon i \to j$ to indicate that an arrow $a \in Q_1$ has tail $i = t(a)$ and head $j = h(a)$. Given an integer $\ell \geq 1$, a non-trivial path of length ℓ in Q is an ordered sequence of arrows

$$p = a_\ell \cdots a_1,$$

such that $h(a_k) = t(a_{k+1})$ for $1 \leq k < \ell$. This path p has tail $t(p) = t(a_1)$, head $h(p) = h(a_\ell)$, and is represented pictorially as follows.

$$\bullet \xleftarrow{a_\ell} \bullet \longleftarrow \cdots \longleftarrow \bullet \xleftarrow{a_1} \bullet$$

For each vertex $i \in Q_0$, e_i is the *trivial path* in Q, with tail and head i, and length 0. A *path* in Q is either a trivial path or a non-trivial path in Q. The path algebra kQ is the associative algebra with underlying vector space

$$kQ = \bigoplus_{\text{paths } p} kp,$$

that is, kQ has a basis consisting of all the paths in Q, with the product pq of two non-trivial paths p and q given by the obvious path concatenation if $t(p) = h(q)$, $pq = 0$ otherwise, $pe_{t(p)} = e_{h(a)}p = p$, $pe_i = e_j p = 0$, for non-trivial paths p and $i, j \in Q_0$ such that $i \neq tp, j \neq hp$, and $e_i e_i = e_i, e_i e_j = 0$ for all $i, j \in Q_0$ such that $i \neq j$. We will always assume that a quiver Q is finite, i.e. its vertex and arrow sets are finite, so kQ has a unit

$$1 = \sum_{i \in Q_0} e_i. \tag{3.3.1}$$

Define vector spaces

$$R_Q = \bigoplus_{i \in Q_0} ke_i, \quad V_Q = \bigoplus_{a \in Q_1} ka.$$

Then $R_Q \subset kQ$ is a semisimple commutative (associative) algebra, because it is the subalgebra spanned by the trivial paths, which are a complete set of orthogonal idempotents of kQ, i.e. $e_i^2 = e_i$, $e_ie_j = 0$ for $i \neq j$, and

$$x = \sum_{i,j \in Q_0} e_jxe_i, \text{ for all } x \in kQ$$

(by (3.3.1)). Furthermore, as V_Q is a vector space with basis consisting of the arrows, it is an R_Q-bimodule with multiplication $e_jae_i = a$ if $a\colon i \to j$ and $e_iae_j = 0$ otherwise, and the path algebra is the tensor algebra of the bimodule V_Q over $R := R_Q$, that is (see Proposition 1.3 in [9]),

$$kQ = \mathrm{T}_R V_Q, \tag{3.3.2}$$

where a path $p = a_\ell \cdots a_1 \in kQ$ is identified with a tensor product $a_\ell \otimes \cdots \otimes a_1 \in \mathrm{T}_R V_Q$.

Let $A = kQ$. It is well known [1] that the decomposition

$$A = \bigoplus_{i \in Q_0} Ae_i,$$

is a decomposition of the A-module ${}_AA$ as a direct sum of pairwise non-isomorphic indecomposable projective A-modules. Note that the vector space underlying Ae_i has a basis consisting of all the paths in Q with tail i. In fact, $\{Ae_i \mid i \in Q_0\}$ is a complete set of indecomposable finitely generated A-modules up to isomorphism (see, for instance, [7]). Furthermore, the evaluation map

$$\mathrm{Hom}_A(Ae_i, M) \xrightarrow{\cong} e_iM\colon f \longmapsto f(e_i) \tag{3.3.3}$$

is a natural isomorphism, for all $i \in Q_0$ and all A-modules M.

Graded quivers and graded path algebras

Graded quivers have turned out to be important objects because we can associate to a (graded) quiver with potential a Ginzburg dg-algebra (see [40]), algebraic structures arising naturally in the geometry of Calabi–Yau manifolds and mirror symmetry. In fact, in this thesis, we require double graded quivers whose definition is based on [67] and [98], §10.3, and the definitions introduced in §3.3.1 for quivers and their modules. Finally, $\mathbb{N} = \mathbb{Z}_{\geq 0}$ will be the set of natural numbers, i.e. the non-negative integers.

Definition 3.3.4. A *graded quiver* is a quiver P together with a map

$$|-|\colon P_1 \longrightarrow \mathbb{N}\colon \quad a \longmapsto |a|$$

that to each arrow a assigns its *weight* $|a|$. The *weight* of the graded quiver is

$$|P| := \max_{a \in P_1} |a|.$$

A *double graded quiver* $\overline{P}$ is obtained from a graded quiver P of weight $|P|$ by adjoining a reverse arrow $a^*\colon j \to i$ for each arrow $a\colon i \to j$ in P and whose weight is $|a^*| = |P| - |a|$.

[1] See e.g. §4.6 of https://www.math.uni-bielefeld.de/~sek/kau/text.html.

From the definition of double graded quiver, observe that if $|P| = 0$, we recover the usual definition of a double quiver (see, for instance, [29]) if we consider it as an $\mathbb{N}$-graded quiver of weight 0. Furthermore, the weight function $|-|\colon P_1 \to \mathbb{N}$ induces a graded structure on the vector space underlying the path algebra kP of the underlying quiver of P, where a trivial path e_i in P has weight $|e_i| = 0$, and a non-trivial path $p = a_\ell \cdots a_1$ in P has weight

$$|p| = |a_1| + \cdots + |a_\ell|.$$

Since $|pq| = |p| + |q|$ for any paths p, q in P such that $t(p) = h(q)$, the path algebra kP is in fact a graded associative algebra, called the *graded path algebra* of P, whose weight is equal to the weight of the graded quiver P.

Let P be a graded quiver, $R := R_P$ the algebra with basis the trivial paths in P, and

$$V_P = \bigoplus_{a \in P_1} ka \tag{3.3.5}$$

the graded R-bimodule with basis consisting of the arrows in P_1, where $a \in P_1$ has weight $|a|$, and multiplications $e_j a e_i = a$ if $i = t(a), j = h(a)$, and $e_i a e_j = 0$ otherwise, for all $a \in P_1$. Recall the construction of the graded tensor algebra of a graded bimodule (see [71, p179]). As in (3.3.2), the graded path algebra kP is the graded tensor algebra

$$kP = \mathrm{T}_R V_P, \tag{3.3.6}$$

where a path $p = a_\ell \cdots a_1 \in kP$ is identified with a tensor product $a_\ell \otimes \cdots \otimes a_1 \in \mathrm{T}_R V_P$.

The graded path algebra kP can be expressed as a graded tensor algebra in another way, using the following two subquivers of P. The *weight 0 subquiver* of P is the (ungraded) quiver Q with vertex set $Q_0 = P_0$, arrow set $Q_1 = \{a \in P_1 \mid |a| = 0\}$, and tail and head maps $t, h\colon Q_1 \to Q_0$ obtained restricting the tail and head maps of P. The *higher-weight subquiver* of P is the graded quiver P^+ with vertex set $P_0^+ = P_0$, arrow set $P_1^+ = \{a \in P_1 \mid |a| > 0\}$, tail and head maps $t, h\colon P_1^+ \to P_0^+$ obtained restricting the tail and head maps of P, and weight function $P_1^+ \to \mathbb{N}$ obtained restricting the weight function of P. Later it will also be useful to consider the graded subquivers $P_{(w)} \subset P$ with vertex set P_0 and arrow set $P_{(w),1}$ consisting of all the arrows $a \in P_1^+$ with weight w, for $0 \leq w \leq |P|$.

In the following lemma, $BaB \subset A$ denotes the B-sub-bimodule of ${}_BA_B$ generated by $a \in A$.

Lemma 3.3.7. *Let $B = kQ$ be the path algebra of Q. Define the graded B-bimodule*

$$M_P := \bigoplus_{a \in P_1^+} BaB. \tag{3.3.8}$$

Then M_P is a finitely generated projective B^{e}-module and the graded path algebra $A = kP$ of P is isomorphic to the graded tensor algebra of M_P over B, i.e. there is a canonical isomorphism

$$A = \mathrm{T}_B\, M_P. \tag{3.3.9}$$

Proof. Let B be an arbitrary ring and A an arbitrary B-algebra. Then the A-bimodule ${}_AA_A$ becomes by pullback a graded B-bimodule, denoted ${}_BA_B$, and

$$(BS_kB)\otimes_B(BS_{k-1}B)\otimes_B\cdots\otimes_B(BS_1B) = BS_kBS_{k-1}B\cdots BS_1B, \tag{3.3.10}$$

for any subsets $S_1,\dots,S_k \subset A$, where the B-sub-bimodules $BS_jB \subset {}_BA_B$ in the left-hand side of (3.3.10), for $1 \le j \le k$, are

$$BS_jB := \left\{ \sum_{i=1}^{\ell} b_is_ib_i',\ \text{where } \ell\in\mathbb{N} \text{ and } b_i,b_i'\in B,\ s_i\in S_j, \text{ for all } 1\le i\le \ell \right\},$$

and, more generally, the right-hand side of (3.3.10) is the B-sub-bimodule of ${}_BA_B$ consisting of sums

$$\sum_{i=1}^{\ell} b_k^{(i)}s_k^{(i)}b_{k-1}^{(i)}s_{k-1}^{(i)}b_{k-2}^{(i)}\cdots b_1^{(i)}s_1^{(i)}b_0^{(i)}, \tag{3.3.11}$$

for all $\ell\in\mathbb{N}$, $b_j^{(i)}\in B$, $s_j^{(i)}\in S_j$ $(1\le i\le\ell,\ 1\le j\le k)$. Furthermore,

Claim 3.3.12. *The tensor product*

$$(b_ks_kb_k')\otimes(b_{k-1}s_{k-1}b_{k-1}')\otimes\cdots\otimes(b_1s_1b_1') \in (BS_kB)\otimes_B(BS_{k-1}B)\otimes_B\cdots\otimes_B(BS_1B) \tag{3.3.13}$$

in the left-hand side of (3.3.10)*, for $b_j\in B, s_j\in S_j$ $(1\le j\le k)$, corresponds to the product*

$$b_ks_kb_k'b_{k-1}s_{k-1}b_{k-1}'\cdots s_1b_1' \in BS_kBS_{k-1}B\cdots BS_1B \tag{3.3.14}$$

in A, in the right-hand side of (3.3.10)*.*

Proof. To prove this, we can start with $k=2$, i.e. showing that for any two subsets $S_1,S_2\subset A$,

$$BS_1B\otimes_B BS_2B = B(S_1BS_2)B, \tag{3.3.15}$$

where in the left-hand side, BS_jB is the B-sub-bimodule of ${}_BA_B$ generated by $S_j\subset A$, i.e.

$$BS_jB := \left\{\sum b_is_ib_i' \text{ for finitely many } b_i,b_i'\in B,\ s_i\in S_j\right\}, \tag{3.3.16}$$

for $j=1,2$, and in the right-hand side of (3.3.15), $S_1BS_2\subset A$ is the subset

$$S_1BS_2 := \left\{\sum s_ib_is_i' \text{ for finitely many } b_i\in B,\ s_i\in S_1, s_i'\in S_2\right\}. \tag{3.3.17}$$

Furthermore, the tensor product $(b_1s_1b'_1)\otimes(b'_2s_2b_2)$ in the left-hand side of (3.3.15) corresponds to the product $b_1s_1b'_1b'_2s_2b_2\in A$ in the right-hand side of (3.3.15), for all $b_1,b'_1,b_2,b'_2\in B, s_1\in S_1, s_2\in S_2$.

To prove (3.3.15), note that $BS_1B\otimes_B BS_2B$ is the quotient of $BS_1B\times BS_2B$ by its B-sub-bimodule N generated by $(t_1b,t_2)-(t_1,bt_2)$ for all $b\in B$ and $t_i\in BS_iB$ $(i=1,2)$, where the B-bimodule structure on $BS_1B\times BS_2B$ is given by $b_1(t_1,t_2)b_2:=(b_1t_1,t_2b_2)$, for all $b_1,b_2\in B$ and $(t_1,t_2)\in BS_1B\times BS_2B$. Note now that the B-bimodule morphism

$$BS_1B\times BS_2B\longrightarrow B(S_1BS_2)B\colon (b_1s_1b'_1,b'_2s_2b_2)\longmapsto b_1s_1b'_1b'_2s_2b_2 \tag{3.3.18}$$

(where $b_1,b'_1,b_2,b'_2\in B$, $s_1\in S_1$, $s_2\in S_2$), vanishes on the B-sub-bimodule N and hence induces a B-bimodule morphism

$$BS_1B\otimes_B BS_2B\longrightarrow B(S_1BS_2)B\colon b_1s_1b'_1\otimes b'_2s_2b_2\longmapsto b_1(s_1(b'_1b'_2)s_2)b_2. \tag{3.3.19}$$

The fact that (3.3.19) is an isomorphism (so we obtain (3.3.15)), follows now because one can readily check that the morphism (3.3.19) has an inverse given by the B-bimodule morphism

(3.3.20)
$$B(S_1BS_2)B\longrightarrow BS_1B\otimes_B BS_2B\colon b_1(s_1bs_2)b_2\longmapsto b_1s_1b\otimes s_2b_2=b_1s_1\otimes bs_2b_2$$

(where $b,b_1,b_2\in B$, $s_1\in S_1$, $s_2\in S_2$). Now (3.3.11) follows from (3.3.15) by induction on $k\geq 0$. □

Suppose now $A=kP$ and $B=kQ$. Then A becomes a B-algebra via the inclusion map $B\to A$, and hence it is also a graded B-bimodule, denoted ${}_BA_B$, as above. In this case,

$$M_P=B(P_1^+)B, \tag{3.3.21}$$

so applying (3.3.10) with $S_j=P_1^+$ for all $1\leq j\leq k$, we obtain

$$M_P^{\otimes_B k}=B\overbrace{(P_1^+)B\cdots B(P_1^+)}^{k}B, \tag{3.3.22}$$

where the left-hand side is the kth tensor power of M_P over B, and the right-hand side contains k copies of P_1^+, with a copy of B inserted between two consecutive copies of P_1^+. Since the paths in Q are a basis of the underlying vector space of B, (3.3.11) and (3.3.22) imply that $M_P^{\otimes_B k}$ has a homogeneous basis $\mathcal{B}_k$ consisting of all the paths in P of the form

$$p=p_ka_kp_{k-1}a_{k-1}p_{k-2}\cdots p_1a_1p_0,$$

where $a_1,\dots,a_k\in P_1^+$ and $p_0,\dots,p_k$ are (possibly trivial) paths in Q such that $h(p_{i-1})=t(a_i)$, $h(a_i)=t(p_i)$, for all $1\leq i\leq k$. Hence the disjoint union $\mathcal{B}=\bigcup_{k\geq 0}\mathcal{B}_k$ is a homogeneous basis of the tensor algebra $\mathrm{T}_B\,M_P=\bigoplus_{k\geq 0}M_P^{\otimes_B k}$.

Now, the above disjoint union is also a partition of the set of all the paths in P in subsets with the same number of arrows in P_1^+, so $\mathcal{B}$ is the set of all the paths in P, and hence it is also a homogeneous basis of the graded path algebra $A = kP$. Therefore one can construct a canonical isomorphism (3.3.9) between the graded vector spaces underlying A and $\mathrm{T}_B\, M_P$ by simply identifying the basis elements of $\mathcal{B}$. One can now readily check that this isomorphism preserves weights and, using the correspondence between the products (3.3.13) and (3.3.14), that the multiplication map in $\mathrm{T}_B\, M_P$ and A are in correspondence by this isomorphism. Thus we have constructed an isomorphism (3.3.9) of graded associative algebras, as required.

To prove that M_P is finitely generated and projective, we first observe that the B-module ${}_B B$, the B^{op}-module B_B and the B^{e}-module ${}_{B^{\mathrm{e}}}B^{\mathrm{e}}$ are given by

(3.3.23)
$$ {}_B B = \bigoplus_{j\in Q_0} Be_j, \quad B_B = \bigoplus_{i\in Q_0} e_i B, \quad {}_{B^{\mathrm{e}}}B^{\mathrm{e}} = (B\otimes B)_{\mathrm{out}} = \bigoplus_{i,j\in Q_0} Be_j\otimes e_i B,$$

by (3.3.1), and hence for each $i, j \in Q_0$,

(3.3.24)
$$Be_j = \bigoplus_{\substack{\text{paths } q \text{ in } Q\\ \text{with } t(q)=j}} kq, \quad e_iB = \bigoplus_{\substack{\text{paths } p \text{ in } Q\\ \text{with } h(p)=i}} kp, \quad Be_j\otimes e_iB = \bigoplus_{\substack{\text{paths } p,q \text{ in } Q\\ \text{with } t(q)=j, h(p)=i}} kq\otimes kp,$$

are respectively a finitely generated projective left B-module, a finitely generated projective right B-module and a finitely generated projective B-bimodule, as they are respectively direct summands of ${}_B B$ and B_B and ${}_{B^{\mathrm{e}}}B^{\mathrm{e}} = (B\otimes B)_{\mathrm{out}}$, by (3.3.23).

Furthermore, for each $a \in P_1^+$, with $i = t(a), j = h(a)$, ${}_B A_B$ has a graded B-sub-bimodule

$$BaB = \bigoplus_{\substack{\text{paths } p,q \text{ in } Q,\\ \text{with } h(p)=i,\, t(q)=j}} k(qap)$$

with a homogeneous basis (as a graded vector space) consisting of paths qap (for paths p, q in Q such that $h(p) = i$, $t(q) = j$), where each such path has weight $|qap| = |a|$. One can now readily check that there is an isomorphism of B-bimodules

$$BaB \xrightarrow{\cong} Be_j\otimes e_iB\colon \quad qap \longmapsto q\otimes p$$

mapping a basis element qap (for paths p, q in Q with $hp = i$, $tq = j$) into a basis element $q\otimes p$. Hence BaB is a finitely generated projective graded B-bimodule, for so is $Be_j\otimes e_iB$.

Therefore

$$M_P := \bigoplus_{a\in P_1^+} BaB = \bigoplus_{a\in P_1^+} \bigoplus_{\substack{\text{paths } p,q \text{ in } Q,\\ \text{with } hp=ta, tq=ha}} k(qap) \subset V_P \tag{3.3.25}$$

is a finitely generated projective graded B-bimodule, and it is actually a graded B-sub-bimodule of V_P, with a homogeneous basis (as a graded vector space) consisting of the paths qap in P that contain exactly one arrow of non-zero weight. $\square$

By Lemma 3.3.7, graded path algebras fit in the general framework described in §3.1. In particular, observe that the non-commutative cotangent sequence for a graded path algebra A acquires the simpler form of (3.1.7). Note also that

$$M_P = \bigoplus_{w=1}^{|P|} M_{P_{(w)}}, \tag{3.3.26}$$

is a graded B-bimodule, where

$$M_{P_{(w)}} = E_w[-w] = \bigoplus_{a\in P_{(w),1}} BaB, \tag{3.3.27}$$

is a finitely generated projective B-bimodule of weight w, for $1 \leq w \leq |P|$, because so are the B-bimodules BaB, as shown in the proof of Lemma 3.3.7.

3.3.2 Differential forms and double derivations for quivers

In this section and the subsequent one, following §3.3.1, we consider a graded quiver P of weight N whose graded path algebra can be written as $A = \mathrm{T}_R V_P$ (see (3.3.6)).

As we saw in §2.2.1, explicitly, the A-bimodule $\Omega^1_R A$ is generated over A by the set of symbols $\{\mathrm{d}a \mid a \in A\}$, which satisfy $\mathrm{d}\,(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1\,\mathrm{d}a_1 + \lambda_2\,\mathrm{d}a_2$ and $\mathrm{d}\,(a_1 \cdot a_2) = a_1(\mathrm{d}a_2) + (\mathrm{d}a_1)a_2$. Next, since in our case A is the path algebra of P, we have the canonical isomorphism $A \otimes_R V \otimes_R A = \Omega^1_R A$ (see Proposition 3.1.1. Finally, observe that $\Omega^1_R A = \bigoplus_{a\in P_1}(Ae_{h(a)})\,\mathrm{d}a(e_{t(a)}A) = \bigoplus_{w\in\mathbb{N}}\bigoplus_{|a|=w}(Ae_{h(a)})\,\mathrm{d}a(e_{t(a)}A)$.

Next, following [96], §6, for $a \in P$, we define the element $\frac{\partial}{\partial a} \in \mathbb{D}\mathrm{er}_R A$, which on $b \in P_1$ acts as

$$\frac{\partial b}{\partial a} = \begin{cases} e_{h(a)} \otimes e_{t(a)} & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \tag{3.3.28}$$

Remark 3.3.29. Note that M. Van den Bergh (see, for instance, [96], Proposition 6.2.2) composes from right to left whereas we are using the *opposite* convention.

3.3.3 Casimir elements

Recall that if F is a finitely generated projective graded A^{op}-module, its *Casimir element* $\mathtt{cas}_F$ is the pre-image of the identity under the canonical isomorphism

$$F \otimes_{(A^{\mathrm{e}})^{\mathrm{op}}} F^{\vee} \longrightarrow \mathrm{End}_{(A^{\mathrm{e}})^{\mathrm{op}}} F,$$

where (following [30], §5.3), given a graded A^{op}-module, $F^\vee := \operatorname{Hom}_{A^{\mathrm{e}}}(F, A^{\mathrm{e}}_{A^{\mathrm{e}}})$ is the graded dual A-bimodule equipped with the graded A-bimodule structure induced by the outer A-bimodule structure on $A \otimes A$. In the following result, we determine the Casimir element for a graded quiver:

Lemma 3.3.30. *Let P be a graded quiver. Then*

$$\mathtt{cas}_{V_P} = \sum_{a \in P_1} \widetilde{a} \otimes a$$

is the element Casimir for the $(R^{\mathrm{e}})^{\mathrm{op}}_Q$-module V_P, where $\widetilde{a} \in V_P^\vee$ is given by

$$\widetilde{a}(b) = \begin{cases} e_{h(a)} \otimes e_{t(a)} & \text{if } a = b \\ 0 & \text{otherwise,} \end{cases}$$

for a homogeneous $b \in P_1$.

Proof. First, observe that V_P is an $(R^{\mathrm{e}})^{\mathrm{op}}_Q$-module using (2.1.5). We have to check that $\mathtt{eval}\left(\sum_{a \in P_1} a \otimes \widetilde{a}\right)(b) = b$ for all homogeneous $b \in P_1$. Therefore, recalling that, by convention, we compose arrows from right to left,

$$\mathtt{eval}\left(\sum_{a \in P_1} a \otimes \widetilde{a}\right)(b) = \sum_{a \in P_1} a \cdot \widetilde{a}(b) = e_{h(b)}\, b\, e_{t(b)} = b \qquad \square$$

The reason that explains our interest in Casimir elements for quivers is that the inverse of the canonical map $\mathtt{eval}\colon V_P^\vee \otimes_{R^{\mathrm{e}}} A^{\mathrm{e}} \longrightarrow \operatorname{Hom}_{A^{\mathrm{e}}}(V_P, {}_{A^{\mathrm{e}}}A^{\mathrm{e}})$ (see (2.1.25b)) is given by

(3.3.31)

$$\kappa\colon \operatorname{Hom}_{A^{\mathrm{e}}}(V_P, {}_{A^{\mathrm{e}}}A^{\mathrm{e}}) \to V_P^\vee \otimes_{R^{\mathrm{e}}} A^{\mathrm{e}} \colon g \mapsto \sum_{a \in P_1} \widetilde{a} \otimes g(a) = \sum_{a \in P_1} (-1)^{\square} g''(a) \otimes \widetilde{a} \otimes g'(a),$$

where we regard V_P as an R^{e}_Q-module and we use the isomorphism $V_P^\vee \otimes_{R^{\mathrm{e}}} A^{\mathrm{e}} \simeq A \otimes_R V_P^\vee \otimes_R A$ (see Remark 3.1.12), $g(a) = g'(a) \otimes (g'')^{\mathrm{op}}(a) \in A^{\mathrm{e}}$ for $a \in P_1$ and $(-1)^{\square} = (-1)^{|g''(a)|(|g'(a)|+N-|a|)}$.

3.3.4 The bi-symplectic form for a graded double quiver

Duals and biduals

Let $\overline{P}$ be a double graded quiver of weight N whose graded path algebra will be denoted by A. Following [30], §8.1, consider a function $\varepsilon\colon \overline{P} \longrightarrow \{\pm 1\}$ taking $\varepsilon(a) = 1$ if $a \in P$ and -1 if $a \in P^* := \overline{P} \setminus P$.

It is familiar that there exist four sensitive ways of defining the dual of an R-bimodule. Nevertheless, as [16] points out, all these can be identified by fixing a *trace* on R (which is a finite-dimensional semisimple algebra over k), that is, a k-linear map $\operatorname{Tr}\colon R \longrightarrow k$ such that the bilinear form $R \otimes R \to k\colon\ (a, b) \longmapsto \operatorname{Tr}(ab)$ is symmetric and non-degenerate.

More precisely, let V be an R-bimodule, $V^* := \operatorname{Hom}(V,k)$ and $V^\vee := \operatorname{Hom}(V, R\otimes R)$. Then the function $\operatorname{Tr}\colon R\to k$ allows us define an isomorphism $B\colon V^*\to V^\vee$ by requiring that for $\psi\in V^*$ and $v\in V$

$$\psi(v) = \operatorname{Tr}((B(\psi)')(v))\operatorname{Tr}((B(\psi)'')(v))$$

Now we consider the graded A-bimodule $V_{\overline{P}}$ and the space of linear maps $V^*_{\overline{P}} := \operatorname{Hom}(V_{\overline{P}}, k)$. Recall that A has a basis $\{a\}_{a\in\overline{P}_1}$ consisting of all the paths in $\overline{P}$ so let $\{\widehat{a}\}_{a\in\overline{P}_1}\subset V^*_{\overline{P}}$ be the dual basis. Then, if $b\in A$

$$\begin{aligned}
\widehat{a}(b) &= \delta_{ab}\\
&= \begin{cases} 1 & \text{if } a=b\\ 0 & \text{otherwise}\end{cases}\\
&= \operatorname{Tr}(\delta_{ab}e_{h(a)})\operatorname{Tr}(e_{t(a)})\\
&= \operatorname{Tr}((\widetilde{a})'(b))\operatorname{Tr}((\widetilde{a})''(b))\\
&= \operatorname{Tr}\left(B(\hat{a})'(b)\right)\operatorname{Tr}\left(B(\hat{a})''(b)\right),
\end{aligned}$$

which implies that $B(\widehat{a}) = \widetilde{a}$ and, consequently,

$$B^{-1}(\widetilde{a}) = \widehat{a} \tag{3.3.32}$$

Now, using the function $\varepsilon(a)$, we define

$$\langle -,-\rangle\colon V_{\overline{P}}\times V_{\overline{P}}\to k\colon (a,b)\mapsto \langle a,b\rangle = \begin{cases}\varepsilon(a) & \text{if } a=b^*\\ 0 & \text{otherwise}\end{cases} = \begin{cases} 1 & \text{if } a=b^*\in P_1\\ -1 & \text{if } a=b^*\in P_1^*\\ 0 & \text{otherwise}\end{cases}$$

It is not difficult to see that $(V_{\overline{P}}, \langle -,-\rangle)$ is a symplectic graded vector space of weight N. Moreover, $V_{\overline{P}}$ has a graded R-bimodule structure and the symplectic form $\langle -,-\rangle$ gives an isomorphism of graded R-bimodules. Inspired in [30], define an isomorphism, where $V_{\overline{P}}$ is regarded as a vector space:

$$\#\colon V^*_{\overline{P}} \xrightarrow{\cong} V_{\overline{P}}[-N]\colon\quad \widehat{a}\longmapsto \varepsilon(a)a^* \tag{3.3.33}$$

The bi-symplectic form on a graded double quiver

Let $\omega := \sum_{a\in P_1} \mathrm{d}a\,\mathrm{d}a^*\in \mathrm{DR}^2_R A$. Then,

Proposition 3.3.34. *The 2-form ω is bi-symplectic of weight N, with N even.*

Proof. This result and its proof are graded versions of [30], Proposition 8.1.1(ii). It is clear that $\mathrm{d}\omega = 0$.To prove the non-degeneracy, the idea is to show that $\iota(\omega)$ coincides with the composition of the following isomorphisms of graded A-bimodules which we shall construct below:

(3.3.35)

$$\mathbb{D}\mathrm{er}_R A \xrightarrow[H]{\simeq} A\otimes_R V^*_{\overline{P}}\otimes_R A \xrightarrow[\mathrm{Id}\otimes\#\otimes\mathrm{Id}]{\simeq} A\otimes_R V_{\overline{P}}\otimes_R A \xrightarrow[G]{\simeq} \Omega^1_R A$$

Next, using (3.3.31), the isomorphism H will be defined as the following composite:

$$\mathbb{D}\mathrm{er}_R A \xrightarrow{G^\vee} (A \otimes_R V_{\overline{P}} \otimes_R A)^\vee \xrightarrow{\kappa} A \otimes_R V_{\overline{P}}^\vee \otimes_R A \xrightarrow{\mathrm{Id} \otimes B^{-1} \otimes \mathrm{Id}} A \otimes_R V_{\overline{P}}^* \otimes_R A$$

where

$$\begin{aligned}(A \otimes_R V_{\overline{P}} \otimes_R A)^\vee &= \mathrm{Hom}_{A^{\mathrm{e}}}(A \otimes_R V_{\overline{P}} \otimes_R A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}})\\ A \otimes_R V_{\overline{P}}^\vee \otimes_R A &= A \otimes_R \mathrm{Hom}_{R^{\mathrm{e}}}(V_{\overline{P}}, {}_{R^{\mathrm{e}}}R^{\mathrm{e}}) \otimes_R A\\ A \otimes_R V_{\overline{P}}^* \otimes_R A &= A \otimes_R \mathrm{Hom}(V_{\overline{P}}, k) \otimes_R A\end{aligned}$$

More concretely, if $\Theta \in \mathbb{D}\mathrm{er}_R A$:

$$\begin{aligned}H(\Theta) &= \left((\mathrm{Id} \otimes B^{-1} \otimes \mathrm{Id}) \circ \kappa \circ G^\vee\right)(\Theta)\\ &= \left(\mathrm{Id} \otimes B^{-1} \otimes \mathrm{Id}\right)\left(\sum_{a \in \overline{P}_1} (-1)^{|\Theta''(a)|(|\Theta'(a)|+N-|a|)} \Theta''(a) \otimes \widetilde{a} \otimes \Theta'(a)\right)\\ &= \sum_{a \in \overline{P}_1} (-1)^{|\Theta''(a)|(|\Theta'(a)|+N-|a|)} \Theta''(a) \otimes \widehat{a} \otimes \Theta'(a)\end{aligned}$$

In conclusion, if $(-1)^{\square} := (-1)^{|\Theta''(a)|(|\Theta'(a)|+N-|a|)}$,

$$\begin{aligned}H\colon \mathbb{D}\mathrm{er}_R A &\xrightarrow{\cong} A \otimes_R V_{\overline{P}}^* \otimes_R A\\ \Theta &\longmapsto \sum_{a \in \overline{P}_1} (-1)^{\square} \Theta''(a) \otimes \widehat{a} \otimes \Theta'(a)\end{aligned} \tag{3.3.36}$$

Using (3.3.33), the isomorphism $\mathrm{Id} \otimes \# \otimes \mathrm{Id}$ can be defined in the following terms :

$$\begin{aligned}\mathrm{Id} \otimes \# \otimes \mathrm{Id}\colon A \otimes_R V_{\overline{P}}^* \otimes_R A &\xrightarrow{\cong} A \otimes_R V_{\overline{P}} \otimes_R A\\ \sum_{a \in \overline{P}_1} (-1)^{\square} \Theta''(a) \otimes \widehat{a} \otimes \Theta'(a) &\longmapsto \sum_{a \in \overline{P}_1} (-1)^{\square} \Theta''(a) \otimes \varepsilon(a) a^* \otimes \Theta'(a)\end{aligned} \tag{3.3.37}$$

Finally, we shall describe the image under the canonical isomorphism wrote in Proposition (3.1.1) for the object obtained so far:

$$\begin{aligned}G\colon A \otimes_R V_{\overline{P}} \otimes_R A &\xrightarrow{\cong} \Omega^1_R A\\ \sum_{a \in \overline{P}_1} (-1)^{\square} \Theta''(a) \otimes \varepsilon(a) a^* \otimes \Theta'(a) &\longmapsto \sum_{a \in \overline{P}_1} (-1)^{\square} \varepsilon(a) \Theta''(a)\, \mathrm{d}a^* \Theta'(a)\end{aligned} \tag{3.3.38}$$

To finish the proof, we have to check that the element obtained in (3.3.38) coincides with the map $\mathbb{D}\mathrm{er}_R A \to \Omega^1_R A\colon \Theta \mapsto \iota_\Theta \omega$. Now, by (2.2.11):

$$\begin{aligned}i_\Theta \omega &= i_\Theta \left(\sum_{a \in P_1} \mathrm{d}a\, \mathrm{d}a^*\right)\\ &= \sum_{a \in P_1} \left((-1)^{|\Theta||a|} i_\Theta(\mathrm{d}a)\, \mathrm{d}a^* - \mathrm{d}a(i_\Theta(\mathrm{d}a^*))\right)\\ &= \sum_{a \in P_1} \left((-1)^{|\Theta||a|} \Theta'(a) \otimes (\Theta''(a)\, \mathrm{d}a^*) - (\mathrm{d}a \Theta'(a^*)) \otimes \Theta''(a^*)\right)\end{aligned} \tag{3.3.39}$$

The last step is to apply $(-)^\circ$ (see (2.2.14)) and m (defined in (2.2.15)) to (3.3.39):

$$\begin{aligned}
\iota_\Theta\omega &= m\circ(i_\Theta\omega)^\circ\\
&= m\circ\left(\sum_{a\in P_1}\Big((-1)^{|\Theta||a|}\Theta'(a)\otimes(\Theta''(a)\,\mathrm{d}a^*)-(\mathrm{d}a\Theta'(a^*))\otimes\Theta''(a^*)\Big)\right)^\circ\\
&= m\circ(\sum_{a\in P_1}(-1)^{|\Theta||a|}(-1)^{|\Theta'(a)|(|\Theta''(a)+N-|a|)}(\Theta''(a)\,\mathrm{d}a^*)\otimes\Theta'(a)-\\
&-(-1)^{|\Theta''(a^*)|(|a|+|\Theta'(a^*)|)}\Theta''(a^*)\otimes(\mathrm{d}a\Theta'(a^*)))
\end{aligned}$$

As, by hypothesis, N is even, in the sign of the first summand, we have that $|\Theta||a|+|\Theta'(a)|(|\Theta''(a)+N-|a|)=|\Theta''(a)|(|\Theta'(a)+N-|a|)$:

$$\begin{aligned}
&=\sum_{a\in P_1}((-1)^{|\Theta''(a)|(|\Theta'(a)+N-|a|)}\Theta''(a)\,\mathrm{d}a^*\;\Theta'(a)-\\
&-(-1)^{|\Theta''(a^*)|(|a|+|\Theta'(a^*)|)}\Theta''(a^*)\,\mathrm{d}a\;\Theta'(a^*))\\
&=\sum_{a\in\overline{P}_1}(-1)^{\square}\varepsilon(a)\Theta''(a)\,\mathrm{d}a^*\Theta'(a)
\end{aligned}$$ $\square$

3.3.5 Restriction Theorem in weight 1 for double graded quivers

Theorem 3.3.40. *Let $\overline{P}$ be a double graded quiver of weight 2 whose graded path algebra shall be denoted by $A:=k\overline{P}$. Consider R as the semisimple finite dimensional algebra with basis the trivial paths in P and let B be the smooth path algebra of the weight 0 subquiver of $\overline{P}$.*

Assume that A is endowed with a a bi-symplectic form $\omega\in\mathrm{DR}^2_R(A)$ of weight 2, and that it can be written as $A=\mathrm{T}_B M$ where M is the graded B-bimodule

$$M:=E_1[-1]\oplus E_2[-2],$$

for finitely generated projective B-bimodules E_1 and E_2. Then the isomorphism $\iota(\omega)\colon\mathbb{D}\mathrm{er}_R A\longrightarrow\Omega^1_R A[-2]$ restricts, in weight 1, to the B-bimodule isomorphism

$$(\iota(\omega))_1\colon E_1^\vee\xrightarrow{\cong}E_1\colon\quad\widetilde{a}\longmapsto\varepsilon(a)a^*,\tag{3.3.41}$$

whose inverse is

$$\flat\colon E_1\xrightarrow{\cong}E_1^\vee\colon\quad a\longmapsto\varepsilon(a)\widetilde{a^*}\tag{3.3.42}$$

Proof. Recall that A can be described as $\mathrm{T}_R V_{\overline{P}}$ and $\mathrm{T}_B M_{\overline{P}}$ (see (3.3.5) and (3.3.25) respectively), by Lemma 3.3.7 in this subsection, we shall consider the following commutative diagram, where we will use H and G which will be defined in (3.3.50) and (3.3.46), respectively:

(3.3.43)
$$\begin{tikzcd}
A \otimes_B M_1^\vee \otimes_B A \arrow[r, hook] \arrow[dr, "\underline{H}", "\simeq"'] & \mathbb{D}\mathrm{er}_R A \arrow[r, "\iota(\omega)", "\simeq"'] \arrow[d, "H", "\simeq"'] & \Omega^1_R A \arrow[r, two heads] & A \otimes_B M_1 \otimes_B A \\
 & A \otimes_R V^*_{\overline{P}} \otimes_R A \arrow[r, "\mathrm{Id}\otimes\#\otimes\mathrm{Id}", "\simeq"'] & A \otimes_R V_{\overline{P}} \otimes_R A \arrow[r, two heads] \arrow[u, "G", "\simeq"'] & A \otimes_R V_1 \otimes_R A \arrow[u, "\underline{G}", "\simeq"']
\end{tikzcd}$$

To shorten notation,

$$V_1 := (V_{\overline{P}})_1, \quad M_w := (M_{\overline{P}})_w, \quad w > 0$$

and

$$M_1^\vee := \mathrm{Hom}_{B^\mathrm{e}}(M_1, {}_{B^\mathrm{e}}B^\mathrm{e}), \quad V_1^* := \mathrm{Hom}(V_1, k), \quad V_1^\vee := \mathrm{Hom}_{R^\mathrm{e}}(V_1, {}_{R^\mathrm{e}}R^\mathrm{e})$$

Now, if $a \in P_1$, observe that $A \otimes_B M_1 \otimes_B A \simeq A \otimes_R V_1 \otimes_R A$ since

$$\begin{aligned} A \otimes_B M_1 \otimes_B A &= \bigoplus_{|a|=1} (A \otimes_B BaB \otimes_B A) \\ &\simeq \bigoplus_{|a|=1} (AaA) \simeq \bigoplus_{|a|=1} (A \otimes_R ka \otimes_R A) \\ &= A \otimes_R V_1 \otimes_R A \end{aligned} \tag{3.3.44}$$

We have to define $\underline{G}$ making commutative the following square in the diagram (3.3.43)

$$\begin{tikzcd}
\Omega^1_R A \arrow[r, two heads, "h"] & A \otimes_B M_1 \otimes_B A \\
A \otimes_R V_{\overline{P}} \otimes_R A \arrow[r, two heads, "h'"] \arrow[u, "G", "\simeq"'] & A \otimes_R V_1 \otimes_R A, \arrow[u, "\underline{G}", "\simeq"']
\end{tikzcd}$$

To do that, we shall use the natural projection $M = \bigoplus_{w>0} M_w \to M_1$ and Proposition 3.1.6; in particular, f and ν (see (3.1.9) and Proposition 3.1.6(ii) enable us to write h. If we consider a generator qap in $A \otimes_R V_{\overline{P}} \otimes_R A$ with $a \in P_1$ with $|a| = 1$, p and q are paths in P, which satisfy natural compatibility conditions: $h(p) = t(a)$ and $h(a) = t(q)$ then

$$\begin{aligned} (\mathtt{pr} \circ \nu \circ f \circ G)(q \otimes a \otimes p) &= (\mathtt{pr} \circ \nu \circ f)(q \,\mathrm{d}a\, p) \\ &= (\mathtt{pr} \circ \nu)((0 \oplus (q \otimes a \otimes p)) \bmod Q) \\ &= \mathtt{pr}(q \otimes a \otimes p) \\ &= q \otimes a \otimes p \end{aligned} \tag{3.3.45}$$

Using this morphism and the fact that h' is the identity when it applies to $q \otimes a \otimes p$ with $|a| = 1$, we define the morphism $\underline{G}$ as an extension of the isomorphism G:

$$\underline{G} \colon A \otimes_R V_1 \otimes_R A \xrightarrow{\cong} A \otimes_B M_1 \otimes_B A \colon \quad q \otimes a \otimes p \longmapsto q \otimes a \otimes p \tag{3.3.46}$$

Thus, $\underline{G}$ is an isomorphism.

Now, we define the isomorphism $\underline{H}$ (obtained from H by restriction). Firstly, in order to obtain

$$M_1^\vee \otimes_{R^e} A^e \hookrightarrow \mathbb{D}\mathrm{er}_R A, \tag{3.3.47}$$

we observe that if $a \in \overline{P}_1$ is such that $|a| = 1$ and $\widetilde{a} \in M_1^\vee$ is the corresponding element in the dual basis, then $\widetilde{a} \otimes 1_{A^e} \in M_1^\vee \otimes_{B^e} A^e$ can be regarded as an element belonging to $V_{\overline{P}}^\vee \otimes_{R^e} A^e$ by (3.3.44) and the natural injection.. So our task is to determine an object in $\mathbb{D}\mathrm{er}_R A$ from the element $\widetilde{a} \otimes 1_{A^e} \in V_{\overline{P}}^\vee \otimes_{R^e} A^e$ using the sequence of isomorphisms

$$\begin{aligned} V_{\overline{P}}^\vee \otimes_{R^e} A^e &\simeq \mathrm{Hom}_{R^e}(V_{\overline{P}}, \mathrm{Hom}_{A^e}(A^e, A^e)) \\ &\simeq \mathrm{Hom}_{R^e}(A^e \otimes_{R^e} V_{\overline{P}}, {}_{A^e}A^e) \\ &= \mathrm{Hom}_{A^e}(\Omega^1_R A, {}_{A^e}A^e) \\ &= \mathbb{D}\mathrm{er}_R A \end{aligned} \tag{3.3.48}$$

which, as A is the graded path algebra of the graded double quiver $\overline{P}$, can be explicitly described in the following way:

- Under the first isomorphism $V_{\overline{P}}^\vee \otimes_{R^e} A^e \simeq \mathrm{Hom}_{R^e}(V_{\overline{P}}, \mathrm{Hom}_{A^e}(A^e, A^e))$, we obtain
$$\widetilde{a} \otimes 1_{A^e} \longmapsto \mathtt{eval}(\widetilde{a} \otimes 1_{A^e})$$
where (see (2.1.25b))
$$\begin{aligned} \mathtt{eval}(\widetilde{a} \otimes 1_{A^e}) \colon V_{\overline{P}} &\xrightarrow{\cong} A^e \\ b &\longmapsto \widetilde{a}(b) * 1_{A^e} \end{aligned}$$

- Under the second isomorphism $\mathrm{Hom}_{R^e}(V_{\overline{P}}, \mathrm{Hom}_{A^e}(A^e, A^e)) \simeq \mathrm{Hom}_{R^e}(A^e \otimes_{R^e} V_{\overline{P}}, {}_{A^e}A^e)$,
$$\mathtt{eval}(\widetilde{a} \otimes 1_{A^e}) \longmapsto f_1$$
given by
$$\begin{aligned} f_1(b) \colon A^e &\xrightarrow{\cong} A^e \\ a_1 \otimes b_1^{\mathrm{op}} &\longmapsto (e_{h(a)} \otimes e_{t(a)})(a_1 \otimes b_1^{\mathrm{op}}) \end{aligned}$$
where $b \in V_{\overline{P}}$.

- Under the third isomorphism in (3.3.48), $\mathrm{Hom}_{R^e}(A^e \otimes_{R^e} V_{\overline{P}}, {}_{A^e}A^e) = \mathrm{Hom}_{A^e}(\Omega^1_R A, {}_{A^e}A^e)$, we write
$$f_1 \longmapsto f_2$$
where
$$\begin{aligned} f_2 \colon A^e \otimes_{R^e} V_{\overline{P}} &\xrightarrow{\cong} A^e \\ (a_1 \otimes b_1^{\mathrm{op}}) \otimes b &\longmapsto (a_1 \otimes b_1^{\mathrm{op}})(e_{h(a)} \otimes e_{t(a)}) * 1_{A^e} \end{aligned}$$

- Finally, under the fourth isomorphism, $\operatorname{Hom}_{A^{\mathrm{e}}}(A^{\mathrm{e}} \otimes_{R^{\mathrm{e}}} V_{\overline{P}}, A^{\mathrm{e}}) \xrightarrow{\cong} \operatorname{Hom}_{A^{\mathrm{e}}}(\Omega^1_R A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}})$,
$$f_2 \longmapsto i_{\frac{\partial}{\partial a}},$$
where, using (3.3.28),
$$\begin{aligned} i_{\frac{\partial}{\partial a}} \colon \Omega^1_R A &\longrightarrow A^{\mathrm{e}} \\ (a_1 \otimes b_1^{\mathrm{op}})\, \mathrm{d}b &\longmapsto \pm(a_1 e_{h(a)}) \otimes (e_{t(a)} b_1) \end{aligned}$$

To sum up, the previous discussion enables us to determine (3.3.47),

$$M_1^{\vee} \otimes_{B^{\mathrm{e}}} A^{\mathrm{e}} \longrightarrow \mathbb{D}\mathrm{er}_R\, A\colon \quad \widetilde{a} \otimes 1_{A^{\mathrm{e}}} \longmapsto \frac{\partial}{\partial a}, \tag{3.3.49}$$

where $\frac{\partial}{\partial a} \in \mathbb{D}\mathrm{er}_R\, A$ was defined in (3.3.28). Now, following (3.3.43), we apply the isomorphism H:

$$\begin{aligned} H\left(\frac{\partial}{\partial a}\right) &= \left(\mathrm{Id} \otimes B^{-1} \otimes \mathrm{Id}\right)\left(e_{t(a)} \otimes \widetilde{a} \otimes e_{h(a)}\right) \\ &= e_{t(a)} \otimes \widehat{a} \otimes e_{h(a)} \end{aligned}$$

Therefore, we are able to write $\underline{H}$:

$$\begin{aligned} \underline{H}\colon M_1^{\vee} \otimes_{B^{\mathrm{e}}} A^{\mathrm{e}} &\longrightarrow A \otimes_R V_{\overline{P}}^{*} \otimes_R A \\ \widetilde{a} \otimes 1_{A^{\mathrm{e}}} &\longmapsto e_{t(a)} \otimes \widehat{a} \otimes e_{h(a)}, \end{aligned} \tag{3.3.50}$$

Again in the diagram (3.3.43), we use the isomorphism $\mathrm{Id} \otimes \# \otimes \mathrm{Id}$ (see (3.3.37)):

$$\begin{aligned} \mathrm{Id} \otimes \# \otimes \mathrm{Id}\colon A \otimes_R V_{\overline{P}}^{*} \otimes_R A &\longrightarrow A \otimes_R V_{\overline{P}} \otimes_R A \\ e_{t(a)} \otimes \widehat{a} \otimes e_{h(a)} &\longmapsto e_{t(a)} \otimes \varepsilon(a) a^{*} \otimes e_{h(a)} \end{aligned} \tag{3.3.51}$$

Finally, to construct an element in $A \otimes_B M_1 \otimes_B A$ from (3.3.51), we apply h' and $\underline{G}$ (which were defined in (3.3.45) and (3.3.46), respectively) to obtain the element
$$e_{t(a)} \otimes \varepsilon(a) a^{*} \otimes e_{h(a)} = 1 \otimes \varepsilon(a) a^{*} \otimes 1 \in A \otimes_B M_1 \otimes_B A.$$

Using (3.3.43), we were able to construct an isomorphism between $A \otimes_B M_1^{\vee} \otimes_B A$ and $A \otimes_B M_1 \otimes_B A$ given by $1 \otimes \widetilde{a} \otimes 1 \mapsto 1 \otimes \varepsilon(a) a^{*} \otimes 1$. If $(-)_1$ denotes the part of weight 1, $(A \otimes_B M_1^{\vee} \otimes_B A)_1 = B \otimes_B M_1^{\vee} \otimes_B A \simeq M_1^{\vee}$ because, by hypothesis, the double graded quiver has weight 2. Similarly, $(A \otimes_B M_1 \otimes_B A) \simeq M_1$. Therefore, we obtain the following isomorphism of B-bimodules:

$$\mathcal{R}\colon E_1^{\vee} \xrightarrow{\cong} E_1\colon \quad \widetilde{a} \longmapsto \varepsilon(a) a^{*}, \tag{3.3.52}$$

whose inverse is

$$\flat\colon E_1 \xrightarrow{\cong} E_1^{\vee}\colon \quad a \longmapsto \varepsilon(a)\widetilde{a^{*}} \tag{3.3.53}$$

□

Chapter 4

Bi-symplectic $\mathbb{N}Q$-algebras of weight 1

D. Roytenberg [81] proved that symplectic $\mathbb{N}Q$-manifolds of weight 1 are in 1-1 correspondence with ordinary Poisson manifolds. In this chapter, we extend this result to the non-commutative setting. Once we review Roytenberg's result in §4.1, we carry out the classification of bi-symplectic tensor $\mathbb{N}$-algebras of weight 1 (see §4.2), which are in 1-1 correspondence with smooth associative algebras. In the last section of the chapter, we introduce the essential notion of *bi-symplectic $\mathbb{N}Q$-algebras* (which can be regarded as the non-commutative analogues of symplectic $\mathbb{N}Q$-manifolds) and in Theorem 4.3.2 we classify them in weight 1: bi-symplectic $\mathbb{N}Q$-algebras of weight 1 are in 1-1 correspondence with double Poisson algebras.

4.1 Symplectic $\mathbb{N}Q$-manifolds of weight 1

In this subsection we will review the classification of symplectic $\mathbb{N}Q$-manifolds carried out by D. Roytenberg in [81], Proposition 4.1, which states that symplectic $\mathbb{N}Q$-manifolds of weight 1 are in 1-1 correspondence with ordinary Poisson manifolds.

4.1.1 Basics on symplectic polynomial $\mathbb{N}$-algebras

Recall that an $\mathbb{N}$-*graded manifold* (an $\mathbb{N}$-*manifold*, for short) $\mathcal{M}$ of weight n and dimension $(p; r_1, ..., r_n)$ is a smooth p-dimensional manifold M endowed with a sheaf $C^\infty(\mathcal{M})$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras, which can locally be written as $C^\infty_U(M)[\xi^1_1, ..., \xi^{r_1}_1, \xi^1_2, ..., \xi^{r_2}_2, ..., \xi^1_n, ..., \xi^{r_n}_n]$ with ξ^j_i of weight i and U is an open subset of M. An $\mathbb{N}Q$-*manifold* $(\mathcal{M}, Q)$ is an $\mathbb{N}$-manifold endowed with an integrable homological vector field Q of weight $+1$ (i.e. $[Q, Q] = 2Q^2 = 0$). A *symplectic $\mathbb{N}Q$-manifold* $(\mathcal{M}, \omega, Q)$ is an $\mathbb{N}Q$-manifold whose homological vector field is compatible with ω in the sense that $L_Q\omega = 0$, where L_Q stands for the Lie derivative along the vector field Q.

In graded geometry, it is crucial that every $\mathbb{N}$-manifold comes equipped with the *graded Euler vector field*:

$$\mathrm{Eu} = \sum_{i=1}^{n} |x^i| x^i \frac{\partial}{\partial x^i}, \tag{4.1.1}$$

where $\{x^i\}_{i=1}^n$ are the coordinates of the $\mathbb{N}$–manifolds and $|x^i|$ denotes the weight of x^i or, equivalently, it is the derivation which acts on homogeneous smooth functions via $E(f) = |f|f$. Consequently, the weights are just the eigenvalues with respect to the action of Eu in (4.1.1).

The Euler vector field Eu acts on all canonical objects (tensors, jets,...) on $\mathcal{M}$ by means of the Lie derivative. As a consequence, these objects acquire weights. In particular, following [81], we are interested in homogeneous symplectic and Poisson structures; a *symplectic structure of weight* k is a closed non-degenerate 2-form ω such that $\mathrm{L}_{\mathrm{Eu}}\omega = k\omega$, where L_{Eu} denotes the Lie derivative along the Euler vector field Eu.

The space of all homogeneous functions of weight k will be denoted by A^k. The graded algebra

$$A = \bigoplus_{k\geq 0} A^k,$$

is the *algebra of polynomial functions on the* $\mathbb{N}$*-manifold*. Observe that the algebra of all smooth functions is a completion of A and the *weight* of the $\mathbb{N}$-manifold is by definition the highest weight of a local coordinate.

In this section, let R be a commutative k-algebra, where k is a field of characteristic zero. A *commutative R-algebra* is a commutative k-algebra B equipped with a unit preserving k-algebra embedding $R \to B$. A *commutative B-algebra* is then a commutative R-algebra A endowed with an algebra homomorphism $B \to A$ compatible with the identity map $R \to R$ (in particular, it is unit preserving). Also, observe that the definition of $\mathbb{Z}$-graded commutative B-algebra is contained in §2.1, adding the hypothesis of commutativity:

Definition 4.1.2. Let B a commutative R-algebra.

(i) A *commutative* $\mathbb{N}$-*algebra* over B (shorthand for 'non-negatively graded algebra') is a $\mathbb{Z}$-graded commutative B-algebra A such that $A^i = 0$ for all $i < 0$. We say $a \in A$ is *homogeneous of weight* $|a| = i$ if $a \in A^i$.

(ii) A *polynomial* $\mathbb{N}$-*algebra* over B is a commutative $\mathbb{N}$-algebra A over B which can be written as a symmetric graded algebra $A = \mathrm{Sym}^\bullet_B M$, for a positively graded B-module M, so $M = \bigoplus_{i\in\mathbb{Z}} M^i$, where $M^i = 0$ for $i \leq 0$.

(iii) We say $m \in M$ is *homogeneous of weight* $|m| = i$ if $m \in M^i$.

(iv) The *weight* of a commutative $\mathbb{N}$-algebra A is $|A| := \min_{S\in\mathcal{G}} \max_{a\in S} |a|$, where the elements of $\mathcal{G}$ are the finite sets of homogeneous generators of A.

Next, we define one of the central objects of this section:

Definition 4.1.3 (Symplectic polynomial algebra of weight N)**.** Let A be a polynomial $\mathbb{N}$-algebra. An element $\omega \in \Omega^2_R A$ is a *symplectic form of weight N* if it has the following properties:

(i) ω is closed for the universal derivation d,

(ii) ω is homogeneous of weight N (i.e. $\mathrm{L}_{\mathrm{Eu}}\omega = N\omega$),

(iii) The map of graded A-modules is an isomorphism:

$$\mathrm{i}(\omega)\colon \operatorname{Der}_R A \longrightarrow \Omega^1_R A[-N]\colon \quad \theta \longmapsto \mathrm{i}_\theta \omega.$$

A polynomial $\mathbb{N}$-algebra over B equipped with a symplectic form of weight N is a *symplectic polynomial* $\mathbb{N}$-*algebra of weight N* over B if in the decomposition $A = \operatorname{Sym}^\bullet_B M$ with M a graded B-bimodule, $M^i = 0$ for $i > N$ and M^j is a finitely generated projective B-module for $0 \le j \le N$.

It is well-known that if $\omega \in \Omega^2_R A$ is a symplectic form of weight N, define the *Hamiltonian vector field* $H_a \in \operatorname{Der}_R A$ corresponding to a homogenous $a \in A$ via

$$\iota_{H_a}\omega = \mathrm{d}a, \tag{4.1.4}$$

and write

$$\{a, b\}_\omega = H_a(b) \in A \tag{4.1.5}$$

Moreover, following [20], recall that a derivation $X \in \operatorname{Der}_R A$ is called *symplectic* if the Lie derivative of ω with respect to X vanishes, i.e. $\mathrm{L}_X\omega = 0$. Symplectic structures of weight $k \neq 0$ impose strong constrains on symplectic polynomial $\mathbb{N}$-algebras as the following result shows:

Lemma 4.1.6 ([81], Lemma 2.2)**.** *Let A be a symplectic polynomial* $\mathbb{N}$-*algebra over B. Then*

(i) *Any homogeneous symplectic form ω on A of weight $k \neq 0$ is exact.*

(ii) *Any homogeneous derivation $X \in \operatorname{Der}_R A$ of weight $l < -k$ preserving ω is Hamiltonian.*

Proof. Both results are applications of Cartan's identity. Since ω is a symplectic form of weight $k \neq 0$, $\mathrm{L}_{\mathrm{Eu}}\omega = k\omega$ and $\mathrm{d}\omega = 0$:

$$k\omega = \mathrm{L}_{\mathrm{Eu}}\omega = \mathrm{d}\mathrm{i}_{\mathrm{Eu}}\omega,$$

and we conclude (i). By definition,

$$0 = \mathrm{L}_X\omega = \mathrm{d}\mathrm{i}_X\omega.$$

Taking $H := \mathrm{i}_{\mathrm{Eu}}\mathrm{i}_X\omega$, we use Cartan's identity:

$$\mathrm{d}H = \mathrm{d}\mathrm{i}_{\mathrm{Eu}}\mathrm{i}_X\omega. = \mathrm{L}_{\mathrm{Eu}}(\mathrm{i}_X\omega) = (m+n)\mathrm{i}_X\omega.$$

and (ii) holds. $\square$

Since $H_a(b) = i_{H_a}(\mathrm{d}b)$, (4.1.5) can be written in the following form:

$$\{a, b\}_\omega = \iota_{H_a}\iota_{H_b}\omega. \tag{4.1.7}$$

The essential point is that $(A, \{-,-\}_\omega)$ has a very remarkable structure:

Definition 4.1.8 (Poisson algebra of weight $-N$, [54])**.** Let A be a graded commutative algebra. We will call A a *Poisson algebra of weight* $-N$ if it is equipped with a graded bilinear map (called the *Poisson bracket of weight* $-N$),

$$\{-,-\}\colon A \otimes A \longrightarrow A$$

of weight $-N$ such that the following identities hold:

(i) (Graded anti-symmetry)

$$\{a, b\} = -(-1)^{(|a|-N)(|b|-N)}\{b, a\}, \tag{4.1.9}$$

(ii) (Graded Leibniz)

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|-N)|b|}b\{a, c\}, \tag{4.1.10}$$

(iii) (Graded Jacobi)

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|-N)|b|}\{b, \{a, c\}\}. \tag{4.1.11}$$

Lemma 4.1.12. *If (A, ω) is a symplectic polynomial $\mathbb{N}$-algebra of weight N, then $(A, \{-,-\}_\omega)$ in (4.1.7) is a Poisson algebra of weight $-N$ on A.*

Proof. In order to determine the weight of $\{-,-\}_\omega$, observe that by (4.1.4), $|H_a| + |\omega| = |a|$ and by (4.1.5), $|\{a, b\}_\omega| = |a| + |b| - |\omega|$. Thus, $\{-,-\}_\omega = -N$. ☐

4.1.2 Classification of symplectic polynomial $\mathbb{N}$-algebras

Let B a smooth commutative R-algebra and (A, ω) a symplectic polynomial $\mathbb{N}$-algebra of weight 1.

By (4.1.7) and Lemma 4.1.12, ω, the symplectic form of weight 1 induces $\{-,-\}_\omega$, a Poisson bracket of weight -1 on A which is determined by the following inclusions

$$\{A^0, A^0\}_\omega = 0, \qquad \{A^1, A^0\}_\omega \subset A^0, \quad \{A^1, A^1\}_\omega \subset A^1, \tag{4.1.13}$$

Proposition 4.1.14 ([81], Proposition 3.1)**.** *Symplectic polynomial $\mathbb{N}$-algebras of weight 1 are in 1-1 correspondence, up to isomorphism, with ordinary smooth commutative algebras, via $B \leftrightarrow (\operatorname{Der}_R B, \Omega)$, where Ω is determined by the commutator of derivations on B.*

Proof. Let (A, ω) be a symplectic polynomial $\mathbb{N}$-algebra of weight 1. By definition, the weight of A cannot exceed $|\omega| = 1$ hence $A = \operatorname{Sym}^{\bullet}_B M$, where $B = A^0$ and $M = A^1 = E[-1]$ for a finitely generated projective B-module E.

By (4.1.13), $\{A^1, A^0\}_\omega \subset A^0$, and we can define the following action of the B-module $A^1 = M[-1]$ on $A^0 = B$ is meaningful

$$\rho \colon A^1 \longrightarrow \operatorname{Hom}_R(B, B) \colon \quad a \longmapsto (H_a = \{a, -\}_\omega \colon B \longrightarrow B)$$

As $\{-,-\}_\omega$ is a Poisson bracket, H_a satisfies

$$H_a(bb') = \{a, bb'\}_\omega = b\{a, b'\}_\omega + \{a, b\}_\omega b' = bH_a(b') + H_a(b)b',$$

for all $b, b' \in B$, so $H_a \in \operatorname{Der}_R B$ and the map $\rho \colon A^1 \to \operatorname{Der}_R B[-1]$ is an isomorphism by the non-degeneracy of $\{-,-\}_\omega$. By (4.1.11), we have the following identity:

$$\begin{aligned} X_{\{a_1,a_2\}_\omega}(b) &= \{\{a_1, a_2\}_\omega, b\}_\omega \\ &= -\left(\{\{b, a_1\}_\omega, a_2\}_\omega + \{a_1, \{b, a_2\}_\omega\}_\omega\right) \\ &= \{a_1, \{a_2, b\}_\omega\}_\omega - \{a_2, \{a_1, b\}_\omega\}_\omega \\ &= [X_{a_1}, X_{a_2}](b), \end{aligned}$$

where $[-,-]$ denotes the commutator of derivations on B. Thus, under the isomorphism ρ, the Poisson bracket on A^1 corresponds to the commutator of derivations on B. $\square$

4.1.3 Classification of symplectic polynomial $\mathbb{N}Q$-algebras of weight 1

This subsection is devoted to prove, in an algebraic formulation, Roytenberg's classification of symplectic polynomial $\mathbb{N}Q$-algebras:

Definition 4.1.15. (i) A *commutative* $\mathbb{N}Q$*-algebra over* B (A, Q) is a commutative $\mathbb{N}$-algebra A over B endowed with a derivation $Q \colon A \to A$ of weight $+1$ which is homological, i.e. such that $[Q, Q] = 0$, where $[-,-]$ is the graded commutator of derivations.

(ii) A *polynomial* $\mathbb{N}Q$*-algebra* over B is a commutative $\mathbb{N}Q$-algebra over B whose underlying commutative $\mathbb{N}$-algebra is a polynomial $\mathbb{N}$-algebra over B.

(iii) A *symplectic* $\mathbb{N}Q$*-algebra over* B *of weight* N (A, Q, ω) is a polynomial $\mathbb{N}Q$-algebra endowed with a symplectic form $\omega \in \Omega^2_R A$ of weight N such that

(a) The underlying polynomial $\mathbb{N}$-algebra is a symplectic polynomial $\mathbb{N}$-algebra of weight N over B, and

(b) the homological derivation Q is symplectic.

Theorem 4.1.16 ([81], Theorem 4.1)**.** *Symplectic $\mathbb{N}Q$-algebras of weight 1 are in 1-1 correspondence with Poisson algebras.*

Proof. Let R be a commutative k-algebra, where k is a field of characteristic zero, and B a smooth commutative R-algebra. Let (A, Q, ω) be a symplectic $\mathbb{N}Q$-algebra of weight 1. In particular, Q is a symplectic homological derivation, that is, $\mathrm{L}_Q\omega = 0$ and $[Q, Q] = 0$. By Lemma 4.1.6(ii), we write

$$Q = H_S = \{S, -\}_\omega,$$

where $S \in A$ and $\{-, -\}_\omega$ is the Poisson bracket of weight -1 induced by ω on A. In fact, by definition, Q has weight $+1$ and the identity for all $b \in B$ $\{\Theta, b\}_\omega = H_\Theta(b) = Q(b)$ yields

$$|S| = |Q| - |\{-, -\}_\omega| = 2.$$

Thus $S \in A^2$. Crucially, in Proposition 4.1.14 we established an isomorphism of Gerstenhaber algebras $(A, \{-, -\}_\omega) \cong (\bigwedge_B \mathbb{D}\mathrm{er}_R B, [-, -])$, where, as usual, $[-, -]$ stands for the Schouten bracket of multi-vector fields. This isomorphism preserves weights so, as S is a Hamiltonian quadratic function over A, it corresponds under this isomorphism to an element $P \in \bigwedge_B^2 \mathrm{Der}_R B$.

By the graded Jacobi identity (4.1.11), we have the identity $H_{\{a,b\}_\omega} = [H_a, H_b]$ which applied to $a = b = S$ gives

$$H_{\{S,S\}_\omega} = [Q, Q]$$

Therefore, if $a \in A$,

$$[Q, Q](a) = H_{\{S,S\}_\omega}(a) = \{\{S, S\}_\omega, a\}_\omega,$$

and the relation $[Q, Q] = 0$ is equivalent to $\{S, S\}_\omega \in B$ because the equation (4.1.4) provides $\{S, S\}_\omega \in R$ but B is a commutative R-algebra. However, $\{S, S\}_\omega$ has weight 3 because $|S| = 2$. Therefore, we conclude that $\{S, S\}_\omega = 0$. Furthermore, the isomorphism ρ exchanges appropriately the brackets $\{-, -\}_\omega$ and $[-, -]$, so $\{S, S\}_\omega = 0$ implies that $[P, P] = 0$, that is, P is a Poisson bivector, as required. $\square$

4.2 Classification of bi-symplectic tensor $\mathbb{N}$-algebras of weight 1

From now on, R is a finite dimensional semisimple k-algebra, where k is a field of characteristic zero, and B is a smooth associative R-algebra.

Theorem 4.2.1. *Bi-symplectic tensor smooth $\mathbb{N}$-algebras of weight 1 are in 1-1 correspondence, up to isomorphism, with smooth associative R-algebras. The* *correspondence assigns to each smooth associative R-algebra B, the pair (A, ω) consisting of the tensor $\mathbb{N}$-algebra*

$$A = \mathrm{T}^*[1]B := \mathrm{T}_B(\mathbb{D}\mathrm{er}_R\, B[-1])$$

and the bi-symplectic form ω determined by the double Schouten–Nijenhuis bracket of the tensor algebra of the B-bimodule of double derivations over B.

Proof. Let (A, ω) be a bi-symplectic tensor $\mathbb{N}$-algebra of weight 1. By definition, the weight of A cannot exceed $|\omega| = 1$. Hence $A = \mathrm{T}_B\, M$, where $B = A^0$ and $M = A^1 = E[-1]$ for a finitely generated projective B-bimodule E. Observe that, under these hypothesis, A is a smooth algebra (see Proposition 2.2.23).

Define a double Poisson bracket on A by $\{\!\{a, b\}\!\}_\omega = H_a(b)$ (2.5.4), where $H_a \in \mathbb{D}\mathrm{er}_R\, A$ is the Hamiltonian double derivation corresponding to $a \in A$. This double Poisson bracket is non-degenerate, because ω is bi-symplectic, and has weight -1, since $|\omega| = 1$, so it satisfies

$$\{\!\{A^0, A^0\}\!\}_\omega = 0, \quad \{\!\{A^1, A^0\}\!\}_\omega \subset (A \otimes A)^0, \quad \{\!\{A^1, A^1\}\!\}_\omega \subset (A \otimes A)^1.$$

By the second identity, we can define the following action of the bimodule $A^1 = M[-1]$ on $A^0 = B$, which is meaningful:

$$\rho\colon A^1 \longrightarrow \mathrm{Hom}_{R^e}(B, B \otimes B)\colon a \longmapsto (H_a = \{\!\{a, -\}\!\}_\omega : B \longrightarrow B \otimes B)$$

As $\{\!\{-, -\}\!\}_\omega$ is a double Poisson bracket, H_a satisfies

$$H_a(fg) = \{\!\{a, fg\}\!\}_\omega = f\,\{\!\{a, g\}\!\}_\omega + \{\!\{a, f\}\!\}_\omega\, g = fH_a(g) + H_a(f)g,$$

for all $f, g \in B$, so $H_a \in \mathbb{D}\mathrm{er}_R\, B$ and the map $\rho\colon A^1 \to \mathbb{D}\mathrm{er}_R\, B[-1]$ is an isomorphism, by Theorem 3.2.2.

Now, Proposition 2.4.6 establishes that $\rho(\{\!\{a, b\}\!\}_\omega) = H_{\{\!\{a,b\}\!\}_\omega} = \{\!\{H_a, H_b\}\!\} = \{\!\{\rho(a), \rho(b)\}\!\}$, where $\{\!\{-, -\}\!\}$ is the canonical double Schouten–Nijenhuis bracket on $\mathrm{T}_B\, \mathbb{D}\mathrm{er}_R\, B$. Thus, under the isomorphism ρ, the double Poisson bracket on A^1 corresponds to the double Schouten–Nijenhuis bracket on the noncommutative cotangent bundle $\mathrm{T}_B\, \mathbb{D}\mathrm{er}_R\, B$ on B. $\square$

4.3 Classification of bi-symplectic $\mathbb{N}Q$-algebras of weight 1

Definition 4.3.1. (i) An *associative $\mathbb{N}Q$-algebra* (A, Q) over B is an associative $\mathbb{N}$-algebra A of weight N over B endowed with a double derivation $Q\colon A \to A \otimes A$ of weight $+1$ which is homological, i.e. such that $\{\!\{Q, Q\}\!\} = 0$, where $\{\!\{-, -\}\!\}$ stands for the double Schouten-Nijenhuis bracket.

(ii) A *tensor $\mathbb{N}Q$-algebra* (A, Q) is an associative $\mathbb{N}Q$-algebra over B whose underlying associative $\mathbb{N}$-algebra is a tensor $\mathbb{N}$-algebra over B.

(iii) A *bi-symplectic* $\mathbb{N}Q$*-algebra of weight* N (A, ω, Q) is a tensor $\mathbb{N}Q$-algebra over B endowed with a bi-symplectic form $\omega \in \mathrm{DR}^2_R(A)$ of weight N such that

  (a) The underlying tensor $\mathbb{N}$-algebra over B is a bi-symplectic tensor $\mathbb{N}$-algebra of weight N over B, and

  (b) the homological double derivation Q is bi-symplectic.

By Proposition 2.3.20, if (A, P) is a differential double Poisson algebra (DDP for short –recall §2.3.3), then A is a double Poisson algebra with double Poisson bracket $\{\!\{-,-\}\!\}_P$. Therefore, for a smooth algebra A, the notions of DDP algebra and double Poisson algebra are equivalent. The following result is the main result of this chapter because it is the non-commutative analogue of [81], Theorem 4.1:

Theorem 4.3.2. *Bi-symplectic* $\mathbb{N}Q$*-algebras of weight 1 are in 1-1 correspondence, up to isomorphism, with double Poisson algebras.*

Proof. Let (A, ω, Q) be a bi-symplectic $\mathbb{N}Q$-algebra of weight 1. In particular, Q is a bi-symplectic homological double derivation, that is, $\mathcal{L}_Q\omega = 0$ and $\{\!\{Q, Q\}\!\} = 0$, where $\{\!\{-,-\}\!\}$ denotes the double Schouten–Nijenhuis bracket. By Lemma 2.5.6(ii), we write

$$Q = \{\!\{S, -\}\!\}_\omega ,$$

where $S \in A$ and $\{\!\{-,-\}\!\}_\omega$ is the weight -1 double Poisson bracket induced by the bi-symplectic form ω on A. In fact, by definition, $|Q| = +1$ and the identity (see (2.4.3))

$$\{\!\{S, a\}\!\}_\omega = H_S(a) = Q(a)$$

for all $a \in A$ yields

$$|S| = |Q| - |\{\!\{-,-\}\!\}_\omega| = 2.$$

Thus, $S \in A^2$. Next, by Proposition 4.2.1, the bi-symplectic tensor $\mathbb{N}$-algebra (A, ω) is isomorphic to the noncommutative cotangent bundle. In other words, we have an isomorphism of double Gerstenhaber algebras $(A, \{\!\{-,-\}\!\}_\omega) \cong (\mathrm{T}_B \mathbb{D}\mathrm{er}_R B, \{\!\{-,-\}\!\})$. This isomorphism preserves weights so, as S is a Hamiltonian quadratic function over A, it corresponds under this isomorphism to an element $P \in (\mathrm{T}_B \mathbb{D}\mathrm{er}_R B)_2$, a two-fold of the tensor algebra of $\mathbb{D}\mathrm{er}_R B$ over B.

Now, the identity $H_{\{\!\{a,b\}\!\}_\omega} = \{\!\{H_a, H_b\}\!\}$ given by Proposition 2.4.6, which applies to $a = b = S$, gives

$$H_{\{\!\{S,S\}\!\}_\omega} = \{\!\{Q, Q\}\!\} .$$

Therefore, if $a \in A$,

$$\{\!\{Q, Q\}\!\}(a) = H_{\{\!\{S,S\}\!\}_\omega}(a) = \{\!\{\{\!\{S, S\}\!\}_\omega , a\}\!\}_\omega .$$

Now, the relation $\{\!\{Q, Q\}\!\} = 0$ is equivalent to $\{\!\{\Theta, \Theta\}\!\}_\omega \in B \otimes B$ because, by (2.4.2), $H_{\{\!\{\Theta,\Theta\}\!\}_\omega} = 0$ implies that $0 = \mathrm{d}\{\!\{\Theta, \Theta\}\!\}_\omega = \mathrm{d}\{\!\{\Theta, \Theta\}\!\}'_\omega \otimes \{\!\{\Theta, \Theta\}\!\}''_\omega + \{\!\{\Theta, \Theta\}\!\}'_\omega \otimes \mathrm{d}\{\!\{\Theta, \Theta\}\!\}''_\omega$, which implies that $\{\!\{\Theta, \Theta\}\!\}'_\omega , \{\!\{\Theta, \Theta\}\!\}''_\omega \in R$. Finally, since B is an associative R-algebra, we conclude that $\{\!\{\Theta, \Theta\}\!\}'_\omega , \{\!\{\Theta, \Theta\}\!\}''_\omega \in B$ as we required and, as a consequence, $|\{\!\!\{\Theta, \Theta\}\!\!\}_\omega| = 0$. However, $\{\!\!\{\Theta, \Theta\}\!\!\}_\omega$ has weight 3 because $|\Theta| = 2$. So $\{\!\!\{\Theta, \Theta\}\!\!\}_\omega = 0$.

Furthermore, by Proposition 2.4.6, the isomorphism ρ exchanges appropriately the brackets $\{\!\!\{-,-\}\!\!\}_\omega$ and $\{\!\!\{-,-\}\!\!\}$, so $\{\!\!\{S, S\}\!\!\}_\omega = 0$ implies that $\{\!\!\{P, P\}\!\!\} = 0$. Consequently, $\{P, P\} = 0$, and hence (B, P) is a DDP. But since B is smooth, so $(B, \{\!\!\{-,-\}\!\!\}_P)$ is a double Poisson algebra, as a consequence of Proposition 2.3.20. □

Chapter 5

Bi-symplectic tensor $\mathbb{N}$-algebras of weight 2

Chapter 5 is somehow the core of this thesis. In §5.1 we sketch a result of D. Roytenberg that can be reformulated more algebraically using Lie–Rinehart algebras (the algebraic structure corresponding to Lie algebroids) as follows: the structure of a symplectic polynomial $\mathbb{N}$-algebra of weight 2 is completely determined by a finitely generated projective B-module E_1 endowed with a symmetric non-degenerate bilinear form $\langle -,- \rangle$ (see [81], Theorem 3.3). In §5.2, if B is a smooth associative algebra, given a bi-symplectic tensor $\mathbb{N}$-algebra of weight 2 over B $A = \mathrm{T}_B(E_1[-1] \oplus E_2[-2]$ where E_1 and E_2 are projective finitely generated B-bimodules, we calculate A^0, A^1 and A^2, the subspaces $A^w \subset A$ of weights 0,1,2, and determine the structure of the double Poisson bracket of weight -2 induced by the bi-symplectic form. §5.4 is devoted to define a non-commutative counterpart of a Lie–Rinehart algebra (see Definition 5.4.1) and to prove in Proposition 5.4.9 that A^2 has this structure.

A key point in our discussion is that E_1 is endowed with a pairing (in the sense of [97], §3.1), whose definition is reviewed in (5.3.3). Using the results of §3.3, we construct a non-degenerate symmetric pairing for double graded quivers (see Lemma 5.3.7). In §5.3.4, we prove that this pairing is compatible, in a suitable sense (see §5.3.4), with certain family of "double covariant differential operators" $\mathbb{D}_a$ introduced in §5.3.2.

In §5.5 we introduce the notion of *double Atiyah algebra* and *metric double Atiyah algebra* $\mathbb{A}\mathrm{t}(E_1)$ which are endowed with brackets (5.5.8), resembling Van den Bergh's double Schouten–Nijenhuis bracket. In Proposition 5.5.10 we prove that $\mathbb{A}\mathrm{t}(E_1)$ is a double Lie–Rinehart algebra. Finally, §5.6 is devoted to prove that a map $\Psi\colon A^2 \to \mathbb{A}\mathrm{t}(E_1)$, defined in (5.6.2) using the "double covariant differential operators", is a map of twisted double Lie–Rinehart algebras (see Proposition 5.6.1). Furthermore, in §5.7 we demonstrate that, in the setting of double graded quivers, Ψ is an isomorphism and, consequently, we conclude that our bi-symplectic tensor $\mathbb{N}$-algebra A over B of weight 2 is completely determined by E_1 together with its non-degenerate symmetric pairing.

5.1 Symplectic polynomial ℕ-algebras of weight 2

In this subsection we reformulate more algebraically a result of Roytenberg [81], Theorem 3.3. In our formulation, it relates pseudo-euclidean modules over a smooth commutative algebra B (that is, finitely generated projective B-modules endowed with a non-degenerate symmetric bilinear form) and symplectic polynomial ℕ-algebras of weight 2, as in Definition 4.1.3.

5.1.1 The symplectic polynomial ℕ-algebra A

We start reformulating Roytenberg's results [81], §3 in the language of commutative algebra introduced in §4.1. Let R be a commutative algebra, B a smooth commutative R-algebra, and E_1 and E_2 finitely generated projective B-modules. Define the smooth graded symmetric ℕ-algebra

$$A := \operatorname{Sym}_B^{\bullet} M,$$

where

$$M := E_1[-1] \oplus E_2[-2].$$

Let $\omega \in \Omega_R^2 A$ be a symplectic form of weight 2. Thus the pair (A, ω) is a symplectic polynomial ℕ-algebra of weight 2. Now,

$$A = \operatorname{Sym}_B^{\bullet} M = \bigoplus_{n \in \mathbb{N}} A^n,$$

where

$$A^n = \bigoplus_{j+2l=n} \bigwedge\nolimits_B^j E_1[-1] \otimes_B \operatorname{Sym}_B^l(E_2[-2]).$$

In particular, as B-modules,

$$A^0 = B, \quad A^1 = E_1, \quad A^2 = \bigwedge\nolimits_B^2 E_1 \oplus E_2. \tag{5.1.1}$$

By Lemma 4.1.12, ω induces a Poisson bracket $\{-,-\}_\omega$ of weight -2 on A, that satisfies the following relations:

$$\begin{aligned} &\{A^0, A^0\}_\omega = \{A^0, A^1\} = 0, \quad \{A^1, A^1\}_\omega \subset A^0, \\ &\{A^2, A^0\}_\omega \subset A^0, \quad \{A^2, A^1\}_\omega \subset A^1, \quad \{A^2, A^2\}_\omega \subset A^2. \end{aligned} \tag{5.1.2}$$

5.1.2 Lie–Rinehart algebras and A^2

4.1.3.a The family of vector fields X

By (5.1.1) and (5.1.2), $\{A^2, B\}_\omega \subset B$, so we can define the map

$$X\colon A^2 \longrightarrow \operatorname{End}_R B\colon \quad a \longmapsto (X_a := \{a, -\}_\omega|_B\colon B \longrightarrow B)$$

Since $\{-,-\}_\omega$ is a Poisson bracket, in particular, it satisfies the graded Leibniz rule (4.1.10) in its second argument, so

$$X_a(b_1b_2) = \{a, b_1b_2\}_\omega = b_1X_a(b_2) + X_a(b_1)b_2,$$

for all $a \in A^2$, $b_1, b_2 \in B$. In other words, $X_a \in \operatorname{Der}_R B$, so the map X can be seen as a "family of vector fields" parametrized by A^2, i.e. a map

$$X\colon A^2 \longrightarrow \operatorname{Der}_R B\colon \quad a \longmapsto (X_a := \{a,-\}_\omega|_B\colon B \longrightarrow B)\,. \tag{5.1.3}$$

4.1.3.b Lie–Rinehart algebras

For sake of completeness, we recall now the definition of Lie–Rinehart algebra (see, for example, [83], Definition B.1):

Definition 5.1.4 (Lie–Rinehart algebra)**.** Let k be a field of characteristic 0, R a commutative k-algebra and B a commutative R-algebra. A *Lie–Rinehart algebra over B* consists of the following data:

(i) A B-module L;

(ii) A B-module map $\rho\colon L \longrightarrow \operatorname{Der}_R B$, called the *anchor*;

(iii) A R-bilinear Lie bracket $\{-,-\}_L\colon L \otimes L \to L$.

These data are required to satisfy the following additional conditions:

(a) $\{a_1, ba_2\}_L = b\{a_1, a_2\}_L + \rho(a_1)(b)a_2$;

(b) $\rho(\{a_1, a_2\}_L) = [\rho(a_1), \rho(a_2)]$,

for all $a_1, a_2 \in L$, $b \in B$, and where $[-,-]$ denotes the commutator of derivations.

Observe that in (iii), we follow our convention (see §2.1) that all unadorned tensor products are over the base field k. Moreover, the Lie bracket is skew-symmetric and it satisfies the Jacobi identity.

Definition 5.1.5. Let $(L, \rho_L, \{-,-\}_L), (L', \rho_{L'}, \{-,-\}_{L'})$ be Lie–Rinehart algebras. We say that $\varphi\colon L \to L'$ is a *map of Lie–Rinehart algebras* if it preserves brackets:

$$\varphi(\{-,-\}_L) = \{\varphi(-), \varphi(-)\}_{L'}.$$

Example 5.1.6. The prototypical example of Lie–Rinehart algebra is $\operatorname{Der}_R B$ with respect to the commutator bracket $[-,-]$ and the identity map $\operatorname{Der}_R B \to \operatorname{Der}_R B$ as the anchor. As Roytenberg points out in [83], Lie–Rinehart algebras form a category and $\operatorname{Der}_R B$ is the terminal object of the category.

4.1.3.b A^2 as a Lie–Rinehart algebra

Proposition 5.1.7. *A^2 is a Lie–Rinehart algebra, with the bracket*

$$\{-,-\} := \{-,-\}_\omega|_{A^2\otimes A^2} \colon A^2 \otimes A^2 \longrightarrow A^2,$$

and the anchor given by the map $X\colon A^2 \to \operatorname{Der}_R B$ in (5.1.3).

Proof. First of all, the bracket is well-defined, because $\{A^2, A^2\}_\omega \subset A^2$ by (5.1.2). Then we have to heck conditions (a) and (b) of Definition 5.1.4. The proof is based on the fact that $(A, \{-,-\}_\omega)$ is a Poisson algebra of weight -2

Condition (a) follows because by the graded Leibniz rule (recall Definition 4.1.8):

$$\{a_1, ba_2\} = b\{a_1, a_2\}_\omega + \{a_1, b\}_\omega a_2 = b\{a_1, a_2\} + \rho(a_1)(b)a_2$$

for all $a_1, a_2 \in A$ and $b \in B$. Similarly, condition (b) follows because by the graded Jacobi identity (4.1.11), given $b \in B$,

$$\begin{aligned}\rho(\{a_1, a_2\})(b') &= \{\{a_1, a_2\}_\omega, b'\}_\omega \\ &= -\left(\{b', \{a_1, a_2\}_\omega\}_\omega + \{a_1, \{b', a_2\}_\omega\}_\omega\right) \\ &= \{a_1, \{a_2, b'\}_\omega\}_\omega - \{a_2, \{a_1, b'\}_\omega\}_\omega \\ &= [\rho(a_1), \rho(a_2)](b')\end{aligned}$$

□

5.1.3 The inner product

4.1.4.a The family of covariant differential operators D

By (5.1.2), we have the relation $\{A^2, A^1\}_\omega \subset A^1$, so we can consider

$$D\colon A^2 \longrightarrow \operatorname{End}_R E_1\colon \quad a \longmapsto \left(D_a := \{a, -\}_\omega|_{E_1} \colon E_1 \longrightarrow E_1\right). \tag{5.1.8}$$

Observe that this map and X in (5.1.3) are closely related to each other as a consequence of the (graded) Leibniz rule:

$$D_a(be) = bD_a(e) + X_a(b)e$$

for all $a \in A^2$, $e \in E_1$, and $b \in B$. We point out that the pair (X, D) is a *derivation* in the sense of [83] or a *covariant differential operator* as was defined in [60].

4.1.4.b The inner product

By (5.1.1) and the inclusion $\{A^1, A^1\}_\omega \subset A^0$ in (5.1.2), we can define

$$\langle -,-\rangle := \{-,-\}_\omega|_{E_1\otimes E_1} \colon E_1 \otimes E_1 \to B$$

In [81], Roytenberg claims the following result:

Lemma 5.1.9. *$\langle -,-\rangle$ is a non-degenerate symmetric B-bilinear form over E_1.*

$\langle -,-\rangle$ will called an *inner product.*

4.1.4.c The preservation of the inner product

The map D in (5.1.8) preserves the inner product $\langle -, - \rangle$ in the sense that

$$X_a(\langle e_1, e_2\rangle) = \langle D_a(e_1), e_2\rangle + \langle e_1, D_a(e_2)\rangle \tag{5.1.10}$$

for all $a \in A^2$ and $e_1, e_2 \in E_1$. This follows by (4.1.11):

$$\begin{aligned} X_a(\langle e_1, e_2\rangle) &= \{a, \{e_1, e_2\}_\omega\}_\omega \\ &= \{\{a, e_1\}_\omega, e_2\}_\omega + \{e_1, \{a, e_2\}_\omega\}_\omega \\ &= \{D_a(e_1), e_2\}_\omega + \{e_1, D_a(e_2)\}_\omega \\ &= \langle D_a(e_1), e_2\rangle + \langle e_1, D_a(e_2)\rangle \end{aligned} \tag{5.1.11}$$

5.1.4 The Atiyah algebra

4.1.5.a Definition of Atiyah algebra

We will use the following algebraic formulation of Atiyah algebroid (see e.g. [51], §2.2, [52], (1.1.3) Example (c) for non-metric variants).

Definition 5.1.12. We define the *metric Atiyah algebra*, $\mathrm{At}(E_1)$ as the set of pairs (X, D), where $X \in \mathrm{Der}_R B$ and $D \in \mathrm{End}_R E_1$ are such that, for all $b \in B$, and $e, e_1, e_2 \in E_1$,

(i) $D(be) = bD(e) + X(b)e$;

(ii) $X(\langle e_1, e_2\rangle) = \langle D(e_1), e_2\rangle + \langle e_1, D(e_2)\rangle$.

4.1.5.b The Atiyah algebra as a Lie–Rinehart algebra

Proposition 5.1.13. $\mathrm{At}(E_1)$ *is a Lie–Rinehart algebra, with bracket given by*

$$[(X_1, D_1), (X_2, D_2)]_{\mathrm{At}} := ([X_1, X_2], [D_1, D_2]),$$

For all $(X_1, D_1), (X_2, D_2) \in \mathrm{At}(E_1)$,*where* $[X_1, X_2]$ *denotes the commutator of derivations, and anchor*

$$\rho\colon \mathrm{At}(E_1) \longrightarrow \mathrm{Der}_R B\colon \quad (X, D) \longmapsto X.$$

Proof. Note that $[(X_1, D_1), (X_2, D_2)]_{\mathrm{At}} \in \mathrm{At}(E_1)$, because

$$\begin{aligned} [D_1, D_2](be) &= D_1(D_2)(be) - D_2(D_1)(be) \\ &= D_1(bD_2(e) + X_2(b)e) - D_2(bD_1(e) + X_1(b)e) \\ &= b(D_1(D_2(e)) - D_2(D_1(e))) + (X_1(X_2(b)) - X_2(X_1(b)))e \\ &= b[D_1, D_2](e) + [X_1, X_2](b)e \end{aligned}$$

and

$$\begin{aligned}
[X_1, X_2]\langle e_1, e_2\rangle &= X_1\left(X_2\langle e_1, e_2\rangle\right) - X_2\left(X_1\langle e_1, e_2\rangle\right) \\
&= X_1\left(\langle D_2(e_1), e_2\rangle + \langle e_1, D_2(e_2)\rangle\right) - \\
&- X_2\left(\langle D_1(e_1), e_2\rangle + \langle e_1, D_1(e_2)\rangle\right) \\
&= \langle D_1(D_2(e_1)), e_2\rangle + \langle e_1, D_1(D_2(e_2))\rangle - \\
&- \langle D_2(D_1(e_1)), e_2\rangle - \langle e_1, D_2(D_1(e_2))\rangle \\
&= \langle [D_1, D_2](e_1), e_2\rangle + \langle e_1, [D_1, D_2](e_2)\rangle
\end{aligned}$$

Condition (a) of Definition 5.1.4 now follows because $D_1(be) = bD_1(e) + \rho(X_1, D_1)(b)e$ for $e \in E$ and $b \in B$, so

$$\begin{aligned}
[D_1, bD_2](e) &= D_1(bD_2(e)) - b(D_2(D_1(e))) \\
&= bD_1(D_2(e)) + D_1(b)D_2(e) - bD_2(D_1(e)) \\
&= (b(D_1 \circ D_2 - D_2 \circ D_1) + X_1(b)D_2)(e) \\
&= b[D_1, D_2](e) + X_1(b)D_2(e)
\end{aligned}$$

Finally, condition (b) of Definition 5.1.4 is a direct consequence of the definition of the anchor:

$$\begin{aligned}
\rho([(X_1, D_1), (X_2, D_2)]_{\mathrm{At}}) &= \rho([X_1, X_2], [D_1, D_2]) \\
&= [X_1, X_2] \\
&= [\rho(X_1, D_1), \rho(X_2, D_2)]
\end{aligned}$$

□

5.1.5 The map ψ

By Proposition 5.1.7 and Proposition 5.1.13, we showed that A^2 and $\mathrm{At}(E_1)$ are Lie–Rinehart algebras. This section is devoted to prove that

Proposition 5.1.14.

$$\begin{aligned}
\psi\colon A^2 &\longrightarrow \mathrm{At}(E_1) \\
a &\longmapsto (X_a, D_a)
\end{aligned} \tag{5.1.15}$$

is a map of Lie–Rinehart algebras.

Lemma 5.1.16. *For all $a_1, a_2 \in A^2$,*

(i) $X_{\{a_1,a_2\}} = [X_{a_1}, X_{a_2}]$,

(ii) $D_{\{a_1,a_2\}} = [D_{a_1}, D_{a_2}]$.

Proof. This result is a consequence of (4.1.11), since $(A^2, \{-,-\}_\omega|_{A^2\otimes A^2})$ is a Lie algebra. Given $b \in B$,

$$\begin{aligned} X_{\{a_1,a_2\}}(b) &= \{\{a_1,a_2\}_\omega, b\}_\omega \\ &= -\left(\{\{b,a_1\}_\omega, a_2\}_\omega + \{a_1,\{b,a_2\}_\omega\}_\omega\right) \\ &= \{a_1,\{a_2,b\}_\omega\}_\omega - \{a_2,\{a_1,b\}_\omega\}_\omega \\ &= [X_{a_1}, X_{a_2}](b) \end{aligned}$$

and (i) follows. Similarly, if $e \in E_1$,

$$D_{\{a_1,a_2\}}(e) = \{\{a_1,a_2\}_\omega, e\}_\omega = \{a_1,\{a_2,e\}_\omega\}_\omega - \{a_2,\{a_1,e\}_\omega\}_\omega = [D_{a_1}, D_{a_2}](e),$$

and we conclude that (ii) holds. □

Proof. (*of Proposition 5.1.14*). This is a direct consequence of Lemma 5.1.16:

$$\begin{aligned} \psi(\{a_1,a_2\}) &= (X_{\{a_1,a_2\}}, D_{\{a_1,a_2\}}) \\ &= ([X_{a_1}, X_{a_2}], [D_{a_1}, D_{a_2}]) \\ &= [(X_{a_1}, D_{a_1}), (X_{a_2}, D_{a_2})]_{\mathrm{At}} \\ &= [\psi(a_1), \psi(a_2)]_{\mathrm{At}} \end{aligned}$$

□

5.1.6 The isomorphism between A^2 and $\mathrm{At}(E_1)$

To show that the symplectic polynomial $\mathbb{N}$-algebra (A,ω) of weight 2 is completely determined by $(E_1, \langle -,- \rangle)$, we will prove that ψ is an isomorphism.

For this purpose, observe that by the definitions of A^2 and $\mathrm{At}(E_1)$ (see Definition 5.1.12), we have the following commutative diagram, where the arrows are short exact sequences:

(5.1.17)
$$\begin{tikzcd} 0 \arrow[r] & \wedge^2_B E_1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d, "\psi"] & E_2 \arrow[r] \arrow[d, "X"] & 0 \\ 0 \arrow[r] & \mathrm{ad}_B(E_1) \arrow[r] & \mathrm{At}(E_1) \arrow[r, "\rho"] & \mathrm{Der}_R B \arrow[r] & 0, \end{tikzcd}$$

where the top (split) short exact sequence corresponds to the direct sum decomposition $A^2 = \bigwedge^2_B E_1 \oplus E_2$ (see (5.1.1)) and by (5.1.10) the *adjoint module* of E_1 is

(5.1.18) $$\mathrm{ad}_B(E_1) := \{D \in \mathrm{End}_B E_1 \mid \langle D(e_1), e_2\rangle + \langle e_1, D(e_2)\rangle = 0,\ \forall e_1, e_2 \in E_1\},$$

In view of this diagram, in this subsection we shall prove that there exists an isomorphism between $\bigwedge^2_B E_1$ and $\mathrm{ad}_B(E_1)$. The idea is that the map $\bigwedge^2_B E_1 \to \mathrm{ad}_B(E_1)$ is obtained by restriction and duality from the well-known isomorphism $E \otimes_B E^* \simeq \mathrm{End}_B E_1$. As $\bigwedge^2_B E_1 \simeq \mathrm{ad}_B(E_1)$ and $E_2 \simeq \mathrm{Der}_R B$ (see [81]), by diagram chase, we can conclude that ψ is an isomorphism as we required.

Proposition 5.1.19. *If E_1 is a finitely generated projective B-module,*

$$\wedge^2_B E_1 \simeq \mathrm{ad}_B(E_1)$$

Proof. As E_1 is a finitely generated projective B-module, the map

$$f\colon E_1^* \otimes_B E_1 \longrightarrow \mathrm{End}_B E_1\colon \quad \alpha\otimes e \longmapsto f(\alpha\otimes e), \tag{5.1.20}$$

where

$$f(\alpha\otimes e)\colon E_1 \longrightarrow E_1\colon \quad e' \longmapsto \alpha(e')e.$$

is an isomorphism. Next, it is well-known that from a symmetric bilinear form, $\langle -,-\rangle\colon E_1\otimes_B E_1 \longrightarrow B$ we can obtain a map $\flat\colon E_1 \longrightarrow E_1^*$. In addition, the bilinear form $\langle -,-\rangle$ is non-degenerate if and only if $\flat$ is an isomorphism. Denote by $\sharp\colon E_1^* \to E_1$ the inverse of $\flat$. The composition of the isomorphisms (5.1.20) and (3.3.42) tensorized by Id_{E_1} yields the following isomorphism:

$$a := f\circ(\flat\otimes \mathrm{Id}_{E_1})\colon E_1\otimes_B E_1 \xrightarrow{\cong} E_1^*\otimes_B E_1 \xrightarrow{\cong} \mathrm{End}_B E_1, \tag{5.1.21}$$

Next, the inverse of the isomorphism (5.1.21) is the composition:

$$b := (\sharp\otimes \mathrm{Id}_{E_1})\circ f^{-1}\colon \mathrm{End}_B E_1 \longrightarrow E_1\otimes_B E_1 \tag{5.1.22}$$

If we denote $E_1\otimes_B E_1$ by W, we can define

$$\tau\colon W\longrightarrow W\colon \quad e_1\otimes e_2 \longmapsto e_2\otimes e_1, \tag{5.1.23}$$

which enables us to introduce

$$W' := \ker\left(\frac{1}{2}(\mathrm{Id}_W+\tau)\right), \quad W'' := \ker\left(\frac{1}{2}(\mathrm{Id}_W+\tau)\right).$$

By the definitions of W' and W'', it is easy to see that $W = W'\oplus W''$. Now, $\bigwedge^2_B E_1 = W/W''$ and consider the canonical projection $p\colon W \longrightarrow W/W'' = \bigwedge^2_B E_1$. Since $W'' = \ker h''$, h'' induces a morphism $j''\colon \bigwedge^2_B E_1 \longrightarrow W$. By construction, firstly, $p\circ j'' = \mathrm{Id}_{\bigwedge^2_B E_1}$ and, secondly, $p|_{W''} = 0$. So, we conclude that $p'\colon W' \to \bigwedge^2_B E_1$ is an isomorphism.

Let $u_i\in W$ and $v_i\in W^*$. Define the elements $w = \sum_i u_i\otimes v_i$ and $T := a(w)\in \mathrm{End}_B E_1$. Hence, $\tau(w) = \sum_i v_i\otimes u_i$. For all $e_1,e_2\in E_1$, and using both (5.1.21) and the symmetry of the inner product,

$$\langle a(\tau(w))(e_1), e_2\rangle = \sum_i \langle u_i,e_1\rangle\langle v_i,e_2\rangle = \langle e_1, T(e_2)\rangle,$$

where we used (5.1.18). Moreover, by definition, $\langle a(w)(e_1),e_2\rangle = \langle T(e_1),e_2\rangle$. So,

$$\langle a(w+\tau(w))(e_1),e_2\rangle = \langle T(e_1),e_2\rangle + \langle e_1, T(e_2)\rangle$$

and $w\in W'$ if and only if $w+\tau(w)=0$ and, by the last identity, it is equivalent to $\langle T(e_1),e_2\rangle + \langle e_1,T(e_2)\rangle = 0$, which is the condition that characterizes $\mathrm{ad}_B(E_1)$ as we wrote in (5.1.18). So, we conclude that $w \in W'$ if and only if $a(w) \in \mathrm{ad}_B(E_1)$.

Finally, consider the map

$$a' \colon W' \longrightarrow \mathrm{ad}_B(E_1) \tag{5.1.24}$$

obtained by restricting (5.1.21), $a \colon W \longrightarrow \mathrm{End}_B E_1$. Since $w \in W'$ if and only if $a(w) \in \mathrm{ad}_B(E_1)$, (5.1.24) is an isomorphism because a so is. Since $p' \colon W' \to \bigwedge_B^2 E_1$ is an isomorphism, we conclude that $\bigwedge_B^2 E_1 \xrightarrow{\cong} \mathrm{ad}_B(E_1)$, as required. □

By diagram chase, we obtain

Theorem 5.1.25. *The map*

$$\psi \colon A^2 \longrightarrow \mathrm{At}(E_1)$$

defined in (5.1.15) *is an isomorphism of Lie–Rinehart algebras.*

This result enables us to conclude that the structure of the symplectic polynomial ℕ-algebra of weight 2, (A, ω), is completely determined by $(E_1, \langle -, - \rangle)$.

5.1.7 Construction of the symplectic form on A

Let R be a commutative k-algebra over a field of characteristic 0, $((B, \omega_B), E_1, \langle -, - \rangle)$ a triple where (B, ω_B) is a smooth symplectic R-algebra, E_1 is a finitely generated projective B-module, and $\langle -, - \rangle \colon E_1 \times E_1 \to B$ is a symmetric non-degenerate bilinear form. This section is devoted to review the construction of a bi-symplectic form from these data which gives rise to a classification of symplectic polynomial ℕ-algebras of weight 2 (see [81], Theorem 3.3).

Roytenberg achieved this using the minimal symplectic realization of $E_1[-1]$. Since it is difficult to adapt this method to our algebraic formulation, we split the Atiyah sequence (5.1.17):

$$0 \longrightarrow \mathrm{ad}_B(E_1) \longrightarrow \mathrm{At}(E_1) \longrightarrow \mathrm{Der}_R B \longrightarrow 0$$

by fixing a linear connection ∇ on E_1 preserving $\langle -, - \rangle$. By definition, $E_2 = \mathrm{Der}_R B$. Then A can be identified with $\mathrm{Sym}_B^\bullet M$, where $M = E_1[-1] \oplus E_2[-2]$.

To construct the symplectic form $\widetilde{\omega} \in (\Omega_R^2 A)_2$, we follow [79]. From the first fundamental exact sequence (see [72]),

$$0 \longrightarrow \Omega_R^1 B \otimes_B A \longrightarrow \mathrm{Der}_R A \longrightarrow M \otimes_B A \longrightarrow 0, \tag{5.1.26}$$

we construct the dual exact sequence which relates $\mathrm{Der}_R A$ to $\mathrm{Der}_R B$:

$$0 \longrightarrow A \otimes_B M^* \longrightarrow \mathrm{Der}_R A \longrightarrow A \otimes_B \mathrm{Der}_R B \longrightarrow 0, \tag{5.1.27}$$

where $M^* := \operatorname{Hom}_B(M, B)$. Using the connection ∇ we can split this short exact sequence

$$\operatorname{Der}_R A \simeq A \otimes_B (M^* \oplus \operatorname{Der}_R B) \tag{5.1.28}$$

By (5.1.28), to construct the symplectic form $\widetilde{\omega} \in (\Omega^2_R A)_2$ or, equivalently, $\mathrm{i}(\widetilde{\omega})\colon \operatorname{Der}_R A \to \Omega^1_R A[-2]$, we can specify its value $\mathrm{i}_{\phi+\nabla_X}$ on $\phi + \nabla_X \in \operatorname{Der}_R A$, where $\phi \in M^*$, $X \in \operatorname{Der}_R B$. As $A = \operatorname{Sym}^\bullet_B M$ is a smooth algebra because B is a smooth R-algebra and M is a finitely generated projective B-module, $\Omega^1_R A \simeq \operatorname{Hom}_A(\operatorname{Der}_R A, A)$. Then,

$$\begin{aligned} \mathrm{i}_{\phi+\nabla_X}\widetilde{\omega}\colon \operatorname{Der}_R A &\longrightarrow A \\ \psi + \nabla_Y &\longmapsto \mathtt{eval}(\mathrm{i}_{\phi+\nabla_X}\widetilde{\omega}, \psi + \nabla_Y), \end{aligned}$$

where $\mathtt{eval}(-,-)$ denotes the usual pairing between one-forms and derivations. Hence, in order to define $\widetilde{\omega}$, it is enough to determine $\widetilde{\omega}(\phi,\psi)$, $\widetilde{\omega}(\nabla_X, \nabla_Y)$ and $\widetilde{\omega}(\nabla_X, \psi)$ for $\phi, \psi \in M^*$ and $X, Y \in \operatorname{Der}_R B$. Nevertheless, it is easy to check that $\widetilde{\omega}(\phi,\psi) = 0$ unless that $\phi, \psi \in E_1^*$ (for instance, if $\phi_1 \in E_1^*$ and $\phi_2 \in E_2^*$ then $\widetilde{\omega}(\phi_1, \phi_2) = A^{-1} = \{0\}$). The 2-form $\widetilde{\omega} \in (\Omega^2_R A)_2$ is defined by

$$\widetilde{\omega}(\nabla_X, \nabla_Y) = \omega_B(X, Y) + \frac{1}{2}\widetilde{R}(X, Y) \tag{5.1.29}$$

$$\widetilde{\omega}(\phi, \psi) = \langle \phi, \psi \rangle, \tag{5.1.30}$$

$$\widetilde{\omega}(\nabla_X, \phi) = 0, \tag{5.1.31}$$

where $\widetilde{R}$ denotes the contraction of the curvature R of ∇ with the inner product $\langle -, - \rangle$. In [79], Theorem 1, M. Rothstein proved that $\widetilde{\omega}$ is a symplectic form. Finally, observe that, by (5.1.30), we have the identity

$$\{e_1, e_2\}_{\widetilde{\omega}} = \langle e_1, e_2 \rangle$$

for all $e_1, e_2 \in E_1$.

5.2 The algebra A

Let R be a semisimple associative k-algebra, where k is a field of characteristic zero, and B a smooth associative R-algebra. Let E_1 and E_2 be projective finitely generated B-bimodules. Define the smooth graded tensor ℕ-algebra:

$$A := \mathrm{T}_B\, M,$$

of the graded B-bimodule

$$M := E_1[-1] \oplus E_2[-2],$$

where the notation $E_i[-i]$ was introduced in §2.1 Let $\omega \in \mathrm{DR}^2_R(A)$ be a bi-symplectic form of weight 2. Thus the pair (A, ω) is a bi-symplectic tensor ℕ-algebra of weight 2 (see Definition 3.2.1). Then

$$A = \bigoplus_{n \in \mathbb{N}} A^n,$$

where

$$A^n = \bigoplus_{n\in\mathbb{N}} \bigoplus_{j+2l=n} \left(E_1[-1]^{\otimes_B j} \otimes E_2[-2]^{\otimes_B l}\right).$$

In particular,

$$A^0 = B, \quad A^1 = E_1, \quad A^2 = (E_1 \otimes_B E_1) \oplus E_2. \tag{5.2.1}$$

By (2.5.4) and Lemma 2.5.5, the bi-symplectic form ω on A determines a double Poisson bracket of weight -2, denoted by $\{\!\{-,-\}\!\}_\omega$. This bracket satisfies the following relations:

$$\begin{aligned}
&\{\!\{A^0, A^0\}\!\}_\omega = \{\!\{A^0, A^1\}\!\}_\omega = 0,\\
&\{\!\{A^1, A^1\}\!\}_\omega \subset (A\otimes A)^0 = B\otimes B,\\
&\{\!\{A^2, A^0\}\!\}_\omega \subset (A\otimes A)^0 = B\otimes B,\\
&\{\!\{A^2, A^1\}\!\}_\omega \subset (A\otimes A)^1 = (E_1\otimes B)\oplus(B\otimes E_1),\\
&\{\!\{A^2, A^2\}\!\}_\omega \subset (A\otimes A)^2.
\end{aligned} \tag{5.2.2}$$

5.3 The pairing

5.3.1 The family of double derivations $\mathbb{X}$

By (5.2.1) and (5.2.2), $\{\!\{A^2, B\}\!\}_\omega \subset B\otimes B$, so we can define the map

$$\mathbb{X}\colon A^2 \longrightarrow \mathrm{Hom}_{R^{\mathrm{e}}}(B, {}_{B^{\mathrm{e}}}B^{\mathrm{e}})\colon \quad a \longmapsto \left(\mathbb{X}_a := \{\!\{a,-\}\!\}_\omega|_B\colon B\longrightarrow B\otimes B\right). \tag{5.3.1}$$

Since $\{\!\{-,-\}\!\}_\omega$ is a double Poisson bracket of weight -2, it satisfies the graded Leibniz rule (2.3.10) in its second argument (with respect to the outer structure), and so

$$\mathbb{X}_a(b_1b_2) = \{\!\{a, b_1b_2\}\!\}_\omega = b_1\{\!\{a,b_2\}\!\}_\omega + \{\!\{a,b_1\}\!\}_\omega b_2 = b_1\mathbb{X}_a(b_2) + \mathbb{X}_a(b_1)b_2,$$

for all $a\in A^2$, and $b_1, b_2\in B$. Therefore $\mathbb{X}_a \in \mathbb{D}\mathrm{er}_R B$, so $\mathbb{X}$ can be seen as a “family of double derivations” parametrized by A^2, i.e. it is a map

$$\begin{aligned}
\mathbb{X}\colon A^2 &\longrightarrow \mathbb{D}\mathrm{er}_R B\\
a &\longmapsto \mathbb{X}_a := \{\!\{a,-\}\!\}_\omega|_B\colon B\longrightarrow B\otimes B.
\end{aligned} \tag{5.3.2}$$

5.3.2 The family of double differential operators $\mathbb{D}$

By (5.2.2), $\{\!\{A^2, A^1\}\!\}_\omega \subset (A\otimes A)^1 = (E_1\otimes B)\oplus(B\otimes E_1)$, so we can define the following map:

$$\begin{aligned}
\mathbb{D}\colon A^2 &\longrightarrow \mathrm{Hom}_{R^{\mathrm{e}}}(E_1, E_1\otimes B\oplus B\otimes E)\\
a &\longmapsto \mathbb{D}_a := \{\!\{a,-\}\!\}_\omega|_{E_1}
\end{aligned} \tag{5.3.3}$$

By the graded Leibniz rule (2.3.10) applied to $\{\!\{-,-\}\!\}_\omega$,

$$\mathbb{D}_a(be) = b\mathbb{D}_a(e) + \mathbb{X}_a(b)e, \tag{5.3.4a}$$

$$\mathbb{D}_a(eb) = \mathbb{D}_a(e)b + e\mathbb{X}_a(b), \tag{5.3.4b}$$

for all $a\in A^2$, $e\in E_1$, $b\in B$; so $\mathbb{D}$ can be seen as a ‘family of double differential operators’.

5.3.3 The pairing

Given an arbitrary associative k-algebra C, Van den Bergh [97], §3.1, defines a *pairing* between two C-bimodules P and Q as a map

$$\langle -,-\rangle \colon P\times Q\to C\otimes C,$$

such that $\langle p,-\rangle$ is linear for the outer bimodule structure on $C\otimes C$ and $\langle -,q\rangle$ is linear for the inner bimodule structure on $C\otimes C$, for all $p\in P$, $q\in Q$. We say that the pairing is *non-degenerate* if P and Q are finitely generated projective C-bimodules and the pairing induces an isomorphism

$$Q\xrightarrow{\cong} P^{\vee}\colon \quad q\longmapsto \langle -,q\rangle,$$

where, as usual, $(-)^{\vee}$ stands for $\mathrm{Hom}_{C^{\mathrm{e}}}(-,{}_{C^{\mathrm{e}}}C^{\mathrm{e}})$ (see §2.1.1)).

Using the third inclusion in (5.2.2), we define

$$\langle -,-\rangle := \{\!\{-,-\}\!\}_{\omega}\,|_{(E_1\otimes E_1)}\colon E_1\otimes E_1\to B\otimes B$$

Recall that a double bracket, in particular, is a double derivation in its second argument for the *outer* bimodule structure on $B\otimes B$ and, correspondingly, it is a double derivation in its first argument for the *inner* bimodule structure on $B\otimes B$. Therefore, since $\langle -,-\rangle$ is the restriction of $\{\!\{-,-\}\!\}_{\omega}$ to $E_1\otimes E_1$, is immediate that $\langle -,-\rangle$ is a pairing. Moreover, for all $e_1,e_2\in E_1$,

$$\langle e_1,e_2\rangle = \{\!\{e_1,e_2\}\!\}_{\omega}\,|_{(E_1\otimes E_1)} = \sigma_{(12)}\,\{\!\{e_2,e_1\}\!\}_{\omega}\,|_{(E_1\otimes E_1)} = \sigma_{(12)}\langle e_2,e_1\rangle$$

where we used (2.3.11) and $|e_1|=|e_2|=1$. Thus we can conclude that $\langle -,-\rangle$ is *symmetric*.

To prove that this pairing is non-degenerate, we will restrict to the set-up of double graded quivers given in §3.3.

$$\flat\colon E_1\longrightarrow E_1^{\vee}\colon a\mapsto \varepsilon(a)\widetilde{a^*}. \tag{5.3.5}$$

We define the following map:

$$\begin{aligned}\langle -,-\rangle\colon E_1\otimes E_1 &\longrightarrow B\otimes B\\ (a,b)&\longmapsto \flat(a)(b)=\varepsilon(a)\widetilde{a^*}(b).\end{aligned} \tag{5.3.6}$$

Now, we can compute $\{\!\{a,b\}\!\}_{\omega} = i_{\frac{\partial}{\partial a}}\iota_{\frac{\partial}{\partial b}}\omega$, using the tools developed in §3.3, where $a,b\in\overline{P}$ are arrows of weight 1, $\frac{\partial}{\partial a},\frac{\partial}{\partial a}\in \mathbb{D}\mathrm{er}_R A$ (see (3.3.28)) and ω is the symplectic form of Proposition 3.3.34:

$$\begin{aligned}
\{\!\{a,b\}\!\}_\omega &= i_{\frac{\partial}{\partial a}} \iota_{\frac{\partial}{\partial b}} \omega \\
&= i_{\frac{\partial}{\partial a}} \left(\sum_{a\in\overline{P}_1} \varepsilon(b^*) e_{h(b)} (\mathrm{d}b) e_{t(b)} \right) \\
&= \sum_{a\in\overline{P}_1} i_{\frac{\partial}{\partial a^*}} (\varepsilon(b^*) e_{h(b)} (\mathrm{d}b) e_{t(b)}) \\
&= \varepsilon(a) \widetilde{a^*}(b) \\
&= \langle a,b\rangle.
\end{aligned}$$

Therefore, we have the following result:

Lemma 5.3.7. *The map $\langle -,-\rangle\colon E_1\times E_1 \to B\otimes B$ defined in (5.3.6) is a non-degenerate symmetric pairing between E_1 and itself and coincides with the restriction to $E_1\otimes E_1$ of $\{\!\{-,-\}\!\}_\omega$.*

Proof. It is clear that $\langle -,-\rangle$ is a pairing. Now, if a,b are arrows of weight 1, $\sigma_{(12)}\langle a,b\rangle = \varepsilon(b) e_{t(b)}\otimes e_{h(b)} = \langle b,a\rangle$. Finally, the non-degeneration is a consequence of Theorem 3.3.40. □

5.3.4 The preservation of the pairing

Since $\mathbb{D}_a \in (A\otimes A)^1$ for all $a\in A^2$, it is useful to extend the pairing (5.3.6) to a map

$$\langle -,-\rangle_L\colon E_1\times (A\otimes A)^1 \longrightarrow (A\otimes A)^1$$

by

$$\langle e, e'\otimes b\rangle_L = \langle e,e'\rangle \otimes b, \quad \langle e, b\otimes e'\rangle_L = 0, \tag{5.3.8}$$

for all $e,e'\in E_1$ and $b\in B$, where we have used $(A\otimes A)^1 = E_1\otimes B\oplus B\otimes E_1$. Then the map $\mathbb{D}$ preserves the pairing in the sense that

$$\sigma_{(123)}\mathbb{X}_a(\langle e_2,e_1\rangle) = \langle e_1, \mathbb{D}_a(e_2)\rangle_L + \sigma_{(132)}\langle e_2, \mathbb{D}_a(e_1)^\circ\rangle_L, \tag{5.3.9}$$

where the map $\mathbb{X}_a$ in (5.3.2) is extended to $A\otimes A$ using the Leibniz rule. This follows from the graded double Jacobi identity:

$$\begin{aligned}
\langle e_1, \mathbb{D}_a(e_2)\rangle_L &= \{\!\{e_1, \{\!\{a,e_2\}\!\}_\omega\}\!\}_{\omega,L} \\
&= \sigma_{(123)} \{\!\{a, \{\!\{e_2,e_1\}\!\}_\omega\}\!\}_{\omega,L} + \sigma_{(132)} \{\!\{e_2, \{\!\{e_1,a\}\!\}_\omega\}\!\}_{\omega,L} \\
&= \sigma_{(123)} \{\!\{a, \{\!\{e_2,e_1\}\!\}_\omega\}\!\}_{\omega,L} - \sigma_{(132)} \left\{\!\!\left\{e_2, \sigma_{(12)} \{\!\{a,e_1\}\!\}_\omega\right\}\!\!\right\}_{\omega,L} \\
&= \sigma_{(123)}\mathbb{X}_a(\langle e_2,e_1\rangle) - \sigma_{(132)}\langle e_2, \mathbb{D}_a(e_1)^\circ\rangle.
\end{aligned}$$

5.4 A^2 and twisted double Lie–Rinehart algebras

5.4.1 Definition of twisted double Lie–Rinehart algebras

According to Van den Bergh's Definition 3.2.1 in [97], a double Lie algebroid over A is an A-bimodule L with a double Poisson bracket of weight -1 on $\mathrm{T}_A L$. However, this definition has two drawbacks for our purposes; firstly, there is no reference to the anchor map, and, secondly, it only considers double Poisson brackets of weight -1. To have a suitable definition of double Lie–Rinehart algebra in our non-commutative setting, we should require the following:

(i) $(A^2, \{\!\{-,-\}\!\}_\omega, \mathbb{X})$ should have this algebraic structure.

(ii) The "double Atiyah algebroid" should be a double Lie–Rinehart algebra.

(iii) We would like to adapt the method of the proof of [96], Proposition 3.5.1, to show a non-commutative version of Proposition 5.1.14 in §5.1.5.

Definition 5.4.1 (Twisted double Lie–Rinehart algebra)**.** Let R be a semisimple associative k-algebra over a field of characteristic zero, and B be an associative R-algebra. A *twisted double Lie–Rinehart algebra over B* consists of the following data:

(i) A B-bimodule N;

(ii) A B-subbimodule $\overline{N} \subset N$ endowed with a symmetric non-degenerate pairing, $\langle -,-\rangle_{\overline{N}}$, called the *appendix*;

(iii) A B-bimodule map $\rho\colon N \to \mathbb{D}\mathrm{er}_R B$, called the *anchor*;

(iv) A R-bilinear bracket

$$\{\!\{-,-\}\!\}_N : N \otimes N \longrightarrow N \otimes B \oplus B \otimes N \oplus \overline{N} \otimes \overline{N},$$

called the *double bracket*. These data are required to satisfy the following additional conditions:

(a) $\{\!\{a_1, a_2\}\!\}_N = -\sigma_{(12)} \{\!\{a_2, a_1\}\!\}_N$

(b) $\{\!\{a_1, ba_2\}\!\}_N = b\{\!\{a_1, a_2\}\!\}_N + \rho(a_1)(b)a_2$

(c) $\{\!\{a_1, a_2 b\}\!\}_N = \{\!\{a_1, a_2\}\!\}_N b + a_2\rho(a_1)(b)$

(d)

$$\begin{aligned} 0 = \{\!\{a_1, \{\!\{a_2, a_3\}\!\}_N\}\!\}_{N,L} &+ \sigma_{(123)} \{\!\{a_2, \{\!\{a_3, a_1\}\!\}_N\}\!\}_{N,L} + \\ &+ \sigma_{(132)} \{\!\{a_3, \{\!\{a_1, a_2\}\!\}_N\}\!\}_{N,L} \end{aligned}$$

(e) $\rho(\{\!\{a_1, a_2\}\!\}_N) = \{\!\{\rho(a_1), \rho(a_2)\}\!\}_{\mathrm{SN}}$.

where $a_1, a_2, a_3 \in N$, $b \in B$, $\{\!\{-,-\}\!\}_{\mathrm{SN}}$ denotes the double Schouten–Nijenhuis bracket (see (2.3.14)). Finally, observe that all products involved are with respect to the outer bimodule structure, and ρ acts by the Leibniz rule on tensor products.

If the appendix is zero, we say that N is a *double Lie–Rinehart algebra*.

Remark 5.4.2. Following [96], §2.3, if $x' \in \mathrm{T}_B N$ and $x = x_1 \otimes \cdots \otimes x_n \in (\mathrm{T}_B N)_n$ then we define

$$\{\!\{x', x\}\!\}_{N,L} = \{\!\{x', x_1\}\!\}_N \otimes x_2 \otimes \cdots \otimes x_n.$$

Example 5.4.3. Using [96], Theorem 3.2.2, it is easy to see that $\mathbb{D}\mathrm{er}_R B$ is a double Lie–Rinehart algebra with respect to the double Schouten–Nijenhuis bracket $\{\!\{-,-\}\!\}_{\mathrm{SN}}$, the identity map $\mathbb{D}\mathrm{er}_R B \longrightarrow \mathbb{D}\mathrm{er}_R B$ as the anchor, and the zero as appendix.

Observe that $\{\!\{a_1, a_2\}\!\}_N$ decomposes as

$$\{\!\{a_1,a_2\}\!\}_N^{l'} \otimes \{\!\{a_1,a_2\}\!\}_N^{l''} + \{\!\{a_1,a_2\}\!\}_N^{r'} \otimes \{\!\{a_1,a_2\}\!\}_N^{r''} + \{\!\{a_1,a_2\}\!\}_N^{m'} \otimes \{\!\{a_1,a_2\}\!\}_N^{m''}, \tag{5.4.4}$$

where

$$\{\!\{a_1,a_2\}\!\}_N^{l'}, \{\!\{a_1,a_2\}\!\}_N^{r''} \in N, \quad \{\!\{a_1,a_2\}\!\}_N^{r'}, \{\!\{a_1,a_2\}\!\}_N^{l''} \in B,$$
$$\{\!\{a_1,a_2\}\!\}_N^{m'}, \{\!\{a_1,a_2\}\!\}_N^{m''} \in \overline{N}.$$

Now, as by hypothesis, the appendix $\overline{N}$ is endowed with a non-degenerate symmetric pairing, we can define the map

$$\mathbb{T}\colon \overline{N} \longrightarrow \mathrm{Hom}_{B^{\mathrm{e}}}(\overline{N}, B \otimes B)\colon \quad \overline{n}_1 \longmapsto \mathbb{T}_{\overline{n}_1} = \langle \overline{n}_1, - \rangle. \tag{5.4.5}$$

Definition 5.4.6. Let $(N, \{\!\{-,-\}\!\}_N, \rho_N)$ and $(N', \{\!\{-,-\}\!\}_{N'}, \rho_{N'})$ be double twisted Lie–Rinehart algebras. We say that $\varphi\colon N \to N'$ is a *map of twisted double Lie–Rinehart algebras* if satisfies

$$\varphi(\{\!\{a_1,a_2\}\!\}_N) = \{\!\{\varphi(a_1), \varphi(a_2)\}\!\}_{N'} \tag{5.4.7}$$

for all $a_1, a_2 \in N$. Then, by convention, in the left-hand side, we extend the map of twisted double Lie–Rinehart algebra in the following way:

$$\begin{aligned}\varphi(\{\!\{a_1,a_2\}\!\}_N) = \varphi(\{\!\{a_1,a_2\}\!\}_N^{l'}) \otimes \{\!\{a_1,a_2\}\!\}_N^{l''} + \mathbb{T}_{\{\!\{a_1,a_2\}\!\}_N^{m'}} \otimes \{\!\{a_1,a_2\}\!\}_N^{m''} +\\ + \{\!\{a_1,a_2\}\!\}_N^{m'} \otimes \mathbb{T}_{\{\!\{a_1,a_2\}\!\}_N^{m''}} + \{\!\{a_1,a_2\}\!\}_N^{r'} \otimes \varphi(\{\!\{a_1,a_2\}\!\}_N^{r''}).\end{aligned}$$

5.4.2 A^2 as a twisted double Lie–Rinehart algebra

As we showed in (5.2.2), $\{\!\{A^2, A^2\}\!\}_\omega \subset (A \otimes A)^2 \subset A \otimes A$, so we can define

$$\{\!\{-,-\}\!\}_{A^2} := \{\!\{-,-\}\!\}_\omega |_{A^2 \otimes A^2}\colon A^2 \otimes A^2 \longrightarrow (A \otimes A)^2. \tag{5.4.8}$$

Proposition 5.4.9. *A^2 is a double Lie-Rinehart algebra, with the bracket $\{\!\{-,-\}\!\}_{A^2}$, the anchor $\mathbb{X}\colon A^2 \to \mathbb{D}\mathrm{er}_R B$, and the appendix given by $(E_1, \langle -,-\rangle$.*

Proof. We have to check that the conditions (a) - (e) in Definition 5.4.1 hold. (b) is an easy consequence of the graded Leibniz rule:

$$\begin{aligned}\{\!\{a_1, ba_2\}\!\}_{A^2} &= b\,\{\!\{a_1,a_2\}\!\}_\omega + \{\!\{a_1,b\}\!\}_\omega\, a_2\\ &= b\,\{\!\{a_1,a_2\}\!\}_\omega + \mathbb{X}_{a_1}(b)a_2\\ &= b\,\{\!\{a_1,a_2\}\!\}_{A^2} + \rho(a_1)(b)a_2\end{aligned}$$

for all $a_1, a_2 \in A^2$, $b \in B$. Similarly, we check (c).

Next, since $\{\!\{-,-\}\!\}_{A^2}$ is the restriction of $\{\!\{-,-\}\!\}_\omega$ to $A^2 \otimes A^2$, and $(A, \{\!\{-,-\}\!\}_\omega)$ is a double Poisson algebra of weight -2, in particular, the Jacobi identity in (d) holds. By the same reason, we prove (a).

Finally, the compatibility in (e) is an application of Proposition 2.4.6 because, $(A, \{\!\{-,-\}\!\}_\omega)$ is a double Poisson bracket of weight -2. □

5.5 The Double Atiyah algebra

5.5.1 The definition of double Atiyah algebra

To simplify the notation, we will write

$$\mathbb{E}\mathrm{nd}_{R^e}(E_1) := \mathrm{Hom}_{R^e}(E_1, (E_1 \otimes B) \oplus (B \otimes E_1)). \tag{5.5.1}$$

If $e \in E$ and $\mathbb{D} \in \mathbb{E}\mathrm{nd}_{R^e}(E_1)$, we will use the following decomposition:

$$\mathbb{D}(e) = \mathbb{D}^l(e) + \mathbb{D}^r(e) = \mathbb{D}^{l'}(e) \otimes \mathbb{D}^{l''}(e) + \mathbb{D}^{r'}(e) \otimes \mathbb{D}^{r''}(e), \tag{5.5.2}$$

where $\mathbb{D}^l(e) \in E_1 \otimes B$, $\mathbb{D}^r(e) \in B \otimes E_1$ (we omit summation symbols), with

$$\mathbb{D}^{l'}(e), \mathbb{D}^{r''}(e) \in E_1, \quad \mathbb{D}^{l''}(e), \mathbb{D}^{r'}(e) \in B.$$

By (5.3.4), we can introduce the following concept:

Definition 5.5.3 (Double Atiyah algebra)**.** Let E be a B-bimodule. We define the *double Atiyah algebra* as the set of pairs $(\mathbb{X}, \mathbb{D})$ with $\mathbb{X} \in \mathbb{D}\mathrm{er}_R B$ and $\mathbb{D} \in \mathbb{E}\mathrm{nd}_{R^e}(E)$ such that,

(i) $\mathbb{D}(be) = b\mathbb{D}(e) + \mathbb{X}(b)e$,

(ii) $\mathbb{D}(eb) = \mathbb{D}(e)b + e\mathbb{X}(b)$,

for all $b \in B$, and $e \in E$.

Since the (projective finitely generated) B-bimodule E_1 is also equipped with a symmetric non-degenerate pairing $\langle -,-\rangle$, constructed in §5.3.3, we should distinguish those double Atiyah algebras which preserve the pairing (see (5.3.9)):

Definition 5.5.4 (Metric double Atiyah algebra)**.** Let $(E, \langle -, -\rangle)$ be a B-bimodule equipped with a symmetric non-degenerate pairing. We define $\mathbb{A}\mathrm{t}(E)$, the *metric double Atiyah algebra* of E_1, as the double Atiyah algebra which, in addition, preserves the pairing in the sense

$$\sigma_{(123)}\mathbb{X}(\langle e_2, e_1\rangle) = \langle e_1, \mathbb{D}(e_2)\rangle_L + \sigma_{(132)}\langle e_2, \mathbb{D}(e_1)^\circ\rangle_L,$$

for all $e_1, e_2 \in E$, $a \in A^2$, $\mathbb{X} \in \mathbb{D}\mathrm{er}_R\, B$ and $\mathbb{D} \in \mathbb{E}\mathrm{nd}_{R^e}(E)$.

5.5.2 The bracket

The following construction of a double bracket on A is inspired by Van den Bergh's [96], §.3.2, double Schouten–Nijenhuis bracket, which is a non-commutative version of the standard Lie bracket of vector fields. From now on, $(\mathbb{X}_1, \mathbb{D}_1), (\mathbb{X}_2, \mathbb{D}_2) \in \mathbb{A}\mathrm{t}(E_1)$. To define a bracket between double differential operators, we will use decomposition (5.5.2):

$$\begin{aligned}
\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_l &:= (\mathbb{D}_1^l \otimes 1_B)\mathbb{D}_2^l + (\mathbb{X}_1 \otimes 1_{E_1})\mathbb{D}_2^r - \Big((1_{E_1} \otimes \mathbb{X}_2)\mathbb{D}_1^l + (1_B \otimes \mathbb{D}_2^r)\mathbb{D}_1^r\Big)\\
&= \mathbb{D}_1^l(\mathbb{D}_2^{l'}) \otimes \mathbb{D}_2^{l''} + \mathbb{X}_1(\mathbb{D}_2^{r'}) \otimes \mathbb{D}_2^{r''} - \Big(\mathbb{D}_1^{l'} \otimes \mathbb{X}_2(\mathbb{D}_1^{l''}) + \mathbb{D}_1^{r'} \otimes \mathbb{D}_2^r(\mathbb{D}_1^{r''})\Big),\\
\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_r &:= (1_B \otimes \mathbb{D}_1^r)\mathbb{D}_2^r + (1_{E_1} \otimes \mathbb{X}_1)\mathbb{D}_2^l - \Big((\mathbb{X}_2 \otimes 1_{E_1})\mathbb{D}_1^r + (\mathbb{D}_2^l \otimes 1_B)\mathbb{D}_1^l\Big)\\
&= \mathbb{D}_2^{r'} \otimes \mathbb{D}_1^r(\mathbb{D}_2^{r''}) + \mathbb{D}_2^{l'} \otimes \mathbb{X}_1(\mathbb{D}_2^{l''}) - \Big(\mathbb{X}_2(\mathbb{D}_1^{r'}) \otimes \mathbb{D}_1^{r''} + \mathbb{D}_2^l(\mathbb{D}_1^{l'}) \otimes \mathbb{D}_1^{l''}\Big),
\end{aligned}$$

Note that $\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_l$ and $\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_r$ depend on $\mathbb{X}_1$ and $\mathbb{X}_2$, but we omit this dependence to simplify the notation. It is immediate that

$$\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_l = -\{\!\{\mathbb{D}_2, \mathbb{D}_1\}\!\}^{\sim}_r,$$

and

$$\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_l(e), \{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_r(e) \in (E_1 \otimes B \otimes B) \oplus (B \otimes E_1 \otimes B) \oplus (B \otimes B \otimes E_1),$$

for all $e \in E_1$.

Following [96], §3.2, we will regard $\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_l$ and $\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_r$ as elements of $\mathbb{E}\mathrm{nd}_{R^e}(E_1) \otimes B$ and $B \otimes \mathbb{E}\mathrm{nd}_{R^e}(E_1)$ respectively. More precisely, we define new brackets:

$$\begin{aligned}
\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_l &= \tau_{(23)} \circ \{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_l,\\
\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_r &= \tau_{(12)} \circ \{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}^{\sim}_r,
\end{aligned} \tag{5.5.5}$$

where $\tau_{(23)}$ and $\tau_{(12)}$ are permutations which act on tensor products of the form $E_1 \otimes B \otimes B$, $B \otimes E_1 \otimes B$, and $B \otimes B \otimes E_1$. For instance,

$$\begin{aligned}
\tau_{(23)} &: E_1 \otimes B \otimes B \longrightarrow E_1 \otimes B \otimes B\colon \quad e \otimes b_1 \otimes b_2 \longmapsto e \otimes b_2 \otimes b_1,\\
\tau_{(23)} &: E_1 \otimes B \otimes B \longrightarrow B \otimes E_1 \otimes B\colon \quad e \otimes b_1 \otimes b_2 \longmapsto b_1 \otimes e \otimes b_2.
\end{aligned}$$

These new brackets can be decomposed as

$$\begin{aligned}\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}_l &= \{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}'_l\otimes\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}''_l\\ \{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}_r &= \{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}'_r\otimes\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}''_r,\end{aligned}\tag{5.5.6}$$

where $\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}'_l, \{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}''_r\in\operatorname{End}_{R^{\mathrm{e}}}(E_1)$ and $\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}''_l, \{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}'_r\in B$. Again, observe that $\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}_l$ and $\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}_r$ depend on $\mathbb{X}_1$ and $\mathbb{X}_2$, but we omit this dependence to simplify the notation.

Lemma 5.5.7.

$$\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}_r = -\{\!\{\mathbb{D}_2,\mathbb{D}_1\}\!\}^\circ_l.$$

Proof. This result is an easy consequence of the identity $\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}^\sim_l = -\{\!\{\mathbb{D}_2,\mathbb{D}_1\}\!\}^\sim_r$ and the application of the permutation $\sigma_{(12)}$ as in [96], (3.2). □

Finally, for all $b_1,b_2\in B$, $\mathbb{X}_1\in\operatorname{\mathbb{D}er}_R B$ and $\mathbb{D}_1,\mathbb{D}_2\in\operatorname{\mathbb{E}nd}_{R^{\mathrm{e}}}(E_1)$, we define

$$\begin{aligned}\{\!\{b_1,b_2\}\!\} &= 0,\\ \{\!\{\mathbb{D}_1,b_1\}\!\} &= \mathbb{X}_1(b_1),\\ \{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\} &= \{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}_l+\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}_r,\end{aligned}\tag{5.5.8}$$

where, as before, to simplify the notation we omit the dependence of $\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}$ on $\mathbb{X}_1$ and $\mathbb{X}_2$, and we consider the right-hand sides as elements of $(\mathrm{T}_B\operatorname{\mathbb{E}nd}_{R^{\mathrm{e}}}(E_1))^{\otimes 2}$. Then the bracket between double differential operators is the unique extension

$$\{\!\{-,-\}\!\}\colon(\mathrm{T}_B\operatorname{\mathbb{E}nd}_{R^{\mathrm{e}}}(E_1))^{\otimes 2}\to(\mathrm{T}_B\operatorname{\mathbb{E}nd}_{R^{\mathrm{e}}}(E_1))^{\otimes 2}$$

of (5.5.8) of weight -1 to the tensor algebra $\mathrm{T}_B\operatorname{\mathbb{E}nd}_{R^{\mathrm{e}}}(E_1)$ satisfying the Leibniz rule.

Finally, by (5.5.8), we define the following natural bracket which acts on elements of $\mathbb{A}\mathrm{t}(E_1)$:

$$[(\mathbb{X}_1,\mathbb{D}_1),(\mathbb{X}_2,\mathbb{D}_2)]_{\mathbb{A}\mathrm{t}} = (\{\!\{\mathbb{X}_1,\mathbb{X}_2\}\!\}_{\mathrm{SN}},\{\!\{\mathbb{D}_1,\mathbb{D}_2\}\!\}),\tag{5.5.9}$$

where $\{\!\{-,-\}\!\}_{\mathrm{SN}}$ denotes the double Schouten–Nijenhuis bracket. It will be called the *double Atiyah bracket.*

5.5.3 The double Atiyah algebra as a double Lie–Rinehart algebra

Proposition 5.5.10. *$\mathbb{A}\mathrm{t}(E_1)$ is a double Lie–Rinehart algebra with the bracket given by the double Atiyah bracket (5.5.9) and anchor*

$$\rho\colon\ \mathrm{At}(E_1)\longrightarrow\operatorname{\mathbb{D}er}_R B\colon\ (\mathbb{X},\mathbb{D})\longmapsto\mathbb{X}.$$

Proof. Our proof is partially based on the proof of Theorem 3.2.2 in [96]. Condition (a) of Definition 5.4.1 is an easy consequence of Lemma 5.5.7 and [96] (3.4). Next, for all $b \in B$ and $e \in E_1$, we have identities such as

$$\begin{aligned} b\mathbb{D}_2(e) &= b\left(\mathbb{D}_2^{l'}(e) \otimes \mathbb{D}_2^{l''}(e) + \mathbb{D}_2^{r'}(e) \otimes \mathbb{D}_2^{r''}\right)(e) \\ &= \mathbb{D}_2^{l'}(e) \otimes b\mathbb{D}_2^{l''}(e) + \mathbb{D}_2^{r'}(e) \otimes b\mathbb{D}_2^{r''}(e). \end{aligned}$$

Then

$$\begin{aligned} \{\!\{\mathbb{D}_1, b\mathbb{D}_2\}\!\}_r(e) &= \tau_{(12)}\{\!\{\mathbb{D}_1, b\mathbb{D}_2\}\!\}_r^{\sim}(e) \\ &= \tau_{(12)}((1 \otimes b \cdot \otimes 1)(\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_r^{\sim}(e) + \mathbb{D}_2^{l'}(e) \otimes \mathbb{X}_1(b)\mathbb{D}_2^{l''}(e) + \\ &+ \mathbb{D}_2^{r'}(e) \otimes \mathbb{X}_1(b)\mathbb{D}_2^{r''}(e))) \\ &= (b \cdot \otimes 1 \otimes 1)\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_r(e) + \mathbb{X}_1(b)\mathbb{D}_2(e) \\ &= b\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_r(e) + \rho(\mathbb{X}_1, \mathbb{D}_1)(b)\, \mathbb{D}_2(e), \end{aligned}$$

and

$$\begin{aligned} \{\!\{\mathbb{D}_1, b\mathbb{D}_2\}\!\}_l(e) &= \tau_{(23)}\{\!\{\mathbb{D}_1, b\mathbb{D}_2\}\!\}_l^{\sim}(e) \\ &= \tau_{(23)}((1 \otimes 1 \otimes b\cdot)\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_l^{\sim}(e) \\ &= (1 \otimes b \cdot \otimes 1)\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_l(e) \\ &= b\{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}_l(e). \end{aligned}$$

This implies condition (b) of Definition 5.4.1. The proof of condition (iii) is similar. The Jacobi identity in (d) can be proven by copying (3.7.1) in [96], because we have the decomposition (5.5.6) and the definition (5.5.8) corresponds to the definition of the double Schouten–Nijenhuis bracket. Finally, (e) is a consequence of the definition of ρ:

$$\begin{aligned} \rho([(\mathbb{X}_1, \mathbb{D}_1), (\mathbb{X}_2, \mathbb{D}_2)]_{\mathbb{A}\mathrm{t}}) &= \rho(\{\!\{\mathbb{X}_1, \mathbb{X}_2\}\!\}_{\mathrm{SN}}, \{\!\{\mathbb{D}_1, \mathbb{D}_2\}\!\}) \\ &= \{\!\{\mathbb{X}_1, \mathbb{X}_2\}\!\}_{\mathrm{SN}} \\ &= \{\!\{\rho(\mathbb{X}_1, \mathbb{D}_1), \rho(\mathbb{X}_2, \mathbb{D}_2)\}\!\}_{\mathrm{SN}} \end{aligned}$$ □

5.6 The map Ψ

Proposition 5.6.1. *The map*

$$\Psi\colon A^2 \longrightarrow \mathbb{A}\mathrm{t}(E_1)a \longmapsto (\mathbb{X}_a, \mathbb{D}_a), \tag{5.6.2}$$

is a map of twisted double Lie–Rinehart algebras.

Proof. We will partially adapt [96], Proposition 3.5.1. We need to prove that

$$\Psi(\{\!\{a_1, a_2\}\!\}_{A^2}) = [\Psi(a_1), \Psi(a_2)]_{\mathbb{A}\mathrm{t}(E_1)}, \tag{5.6.3}$$

for all $a_1, a_2 \in A^2$, by Definition 5.4.6. More precisely, if we denote $\{\!\{-,-\}\!\}_{A^2}$ by $\{\!\{-,-\}\!\}$, we have to prove the following identities:

$$\mathbb{X}_{\{\!\{a_1,a_2\}\!\}} = \{\!\{\mathbb{X}_{a_1}, \mathbb{X}_{a_2}\}\!\}\,, \tag{5.6.4a}$$

$$\mathbb{D}_{\{\!\{a_1,a_2\}\!\}} = \{\!\{\mathbb{D}_{a_1}, \mathbb{D}_{a_2}\}\!\}\,. \tag{5.6.4b}$$

Observe that since $\{\!\{a_1, a_2\}\!\} \in (A \otimes A)^2$, we can apply the conventions developed in §5.4.1 (in particular, (5.4.4) and (5.4.5)) and, in this way, the left-hand sides of (5.6.4) are meaningful.

As $(A, \{\!\{-,-\}\!\}_\omega)$ is a double Poisson algebra of weight -2, (5.6.4a) is a consequence of Proposition 2.4.6. To prove (5.6.4b), we will use the double Jacobi identity

$$0 = \{\!\{a_1, \{\!\{a_2, e\}\!\}_\omega\}\!\}_{\omega,L} + \tau_{(123)}\{\!\{a_2, \{\!\{e, a_1\}\!\}_\omega\}\!\}_{\omega,L} + \tau_{(132)}\{\!\{e, \{\!\{a_1, a_2\}\!\}_{A^2}\}\!\}_{\omega,L}\,, \tag{5.6.5}$$

where $e \in E_1 \subset A$. By (5.5.2), the first summand in (5.6.5) is

$$\begin{aligned}
\{\!\{a_1, \{\!\{a_2, e\}\!\}_\omega\}\!\}_{\omega,L} &= \{\!\{a_1, \mathbb{D}_{a_2}(e)\}\!\}_{\omega,L}\\
&= \left\{\!\!\left\{a_1, \mathbb{D}^{l'}_{a_2}(e)\right\}\!\!\right\}_\omega \otimes \mathbb{D}^{l''}_{a_2}(e) + \left\{\!\!\left\{a_1, \mathbb{D}^{r'}_{a_2}(e)\right\}\!\!\right\}_\omega \otimes \mathbb{D}^{r''}_{a_2}(e)\\
&= (\mathbb{D}_{a_1} \otimes 1_B)\mathbb{D}^{l}_{a_2}(e) + (\mathbb{X}_{a_1} \otimes 1_{E_1})\mathbb{D}^{r}_{a_2}(e),
\end{aligned}$$

and the second summand is

$$\begin{aligned}
\{\!\{a_2, \{\!\{e, a_1\}\!\}_\omega\}\!\}_{\omega,L} &= -\{\!\{a_2, (\mathbb{D}_{a_1}(e))^\circ\}\!\}_{\omega,L}\\
&= -\left(\left\{\!\!\left\{a_2, \mathbb{D}^{l''}_{a_1}(e)\right\}\!\!\right\}_\omega \otimes \mathbb{D}^{l'}_{a_1}(e) + \left\{\!\!\left\{a_2, \mathbb{D}^{r''}_{a_1}(e)\right\}\!\!\right\}_\omega \otimes \mathbb{D}^{r'}_{a_1}(e)\right)\\
&= -\left((\mathbb{X}_{a_2} \otimes 1_{E_1})(\mathbb{D}^{l}_{a_1}(e))^\circ + (\mathbb{D}_{a_2} \otimes 1_B)(\mathbb{D}^{r}_{a_1}(e))^\circ\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\tau_{(123)}\{\!\{a_2, \{\!\{e, a_1\}\!\}_\omega\}\!\}_{\omega,L} &= -\tau_{(123)}(((\mathbb{X}_{a_2} \otimes 1_{E_1})\sigma_{(12)}\mathbb{D}^{l}_{a_1} + (\mathbb{D}_{a_2} \otimes 1_B)\sigma_{(12)}\mathbb{D}^{r}_{a_1})(e))\\
&= -\tau_{(123)}\tau_{(132)}\left(\left(\mathbb{D}^{l'}_{a_1} \otimes \mathbb{X}_{a_2}(\mathbb{D}^{l''}_{a_1}) + \mathbb{D}^{r'}_{a_1} \otimes \mathbb{D}_{a_2}(\mathbb{D}^{r''}_{a_1}\right)(e)\right)\\
&= -\left((1_{E_1} \otimes \mathbb{X}_{a_2})\mathbb{D}^{l}_{a_1}(e) + (1_B \otimes \mathbb{D}_{a_2})\mathbb{D}^{r}_{a_1}(e)\right).
\end{aligned}$$

Next, we calculate the third summand; combining (5.4.5) and (**??**),

$$\{\!\{e, \{\!\{a_1, a_2\}\!\}_{A^2}\}\!\}_{\omega,L} =$$

$$\begin{aligned}
&= -\left(\left\{\!\!\left\{\{\!\{a_1,a_2\}\!\}^{l'}, e\right\}\!\!\right\}^\circ_\omega \otimes \{\!\{a_1,a_2\}\!\}^{l''} + \left\{\!\!\left\{\{\!\{a_1,a_2\}\!\}^{m'}, e\right\}\!\!\right\}^\circ_\omega \otimes \{\!\{a_1,a_2\}\!\}^{m''}\right),\\
&= -\left(\left(\mathbb{D}_{\{\!\{a_1,a_2\}\!\}^{l'}}(e)\right)^\circ \otimes \{\!\{a_1,a_2\}\!\}^{l''} + \left(\mathbb{T}_{\{\!\{a_1,a_2\}\!\}^{m'}}(e)\right)^\circ \otimes \{\!\{a_1,a_2\}\!\}^{m''}\right),
\end{aligned}$$

where we used the fact that $\{\!\{a_1,a_2\}\!\}^{r'} \in B$ and, consequently, $\left\{\!\!\left\{e, \{\!\{a_1,a_2\}\!\}^{r'}\right\}\!\!\right\} = 0$ because $\{\!\{-,-\}\!\}$ is a double Poisson bracket of weight -2. Finally, since $\tau_{(12)}$ and $\tau_{(23)}$ act on triple tensors (e.g. $E_1 \otimes B \otimes B$),

$$\tau_{(132)}\{\!\{e, \{\!\{a_1, a_2\}\!\}\}\!\}_L =$$

$$= -\tau_{(132)}\tau_{(12)}\left(\mathbb{D}_{\{\!\{a_1,a_2\}\!\}^{l'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{l''}+\mathbb{T}_{\{\!\{a_1,a_2\}\!\}^{m'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{m''}\right)$$
$$= -\tau_{(23)}\left(\mathbb{D}_{\{\!\{a_1,a_2\}\!\}^{l'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{l''}+\mathbb{T}_{\{\!\{a_1,a_2\}\!\}^{m'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{m''}\right).$$

Combining the previous expressions, we obtain

(5.6.6)
$$\begin{aligned}0 &= \{\!\{a_1,\{\!\{a_2,e\}\!\}_\omega\}\!\}_{\omega,L}+\tau_{(123)}\{\!\{a_2,\{\!\{e,a_1\}\!\}_\omega\}\!\}_{\omega,L}+\tau_{(132)}\{\!\{e,\{\!\{a_1,a_2\}\!\}_{A^2}\}\!\}_{\omega,L},\\ &= \{\!\{\mathbb{D}_{a_1},\mathbb{D}_{a_2}\}\!\}_l^{\sim}-\tau_{(23)}(\mathbb{D}_{\{\!\{a_1,a_2\}\!\}^{l'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{l''}+\mathbb{T}_{\{\!\{a_1,a_2\}\!\}^{m'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{m''})\\ &= \{\!\{\mathbb{D}_{a_1},\mathbb{D}_{a_2}\}\!\}_l(e)-\left(\mathbb{D}_{\{\!\{a_1,a_2\}\!\}^{l'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{l''}+\mathbb{T}_{\{\!\{a_1,a_2\}\!\}^{m'}}(e)\otimes\{\!\{a_1,a_2\}\!\}^{m''}\right).\end{aligned}$$

Finally, by Lemma 5.5.7, we get

$$\begin{aligned}-\{\!\{a_2,a_1\}\!\}^{l''}\otimes\mathbb{D}_{\{\!\{a_2,a_1\}\!\}^{l'}}(e) &= -\{\!\{a_2,a_1\}\!\}^{l''}\otimes\left\{\!\!\left\{\{\!\{a_2,a_1\}\!\}^{l'},e\right\}\!\!\right\}_\omega\\ &= -\left\{\!\!\left\{\{\!\{a_2,a_1\}\!\}^{l''}\otimes\{\!\{a_2,a_1\}\!\}^{l'},e\right\}\!\!\right\}_{\omega,R}\\ &= \{\!\{\{\!\{a_1,a_2\}\!\}^{\circ},e\}\!\}_{\omega,R}\\ &= \{\!\{a_1,a_2\}\!\}^{r'}\otimes\left\{\!\!\left\{\{\!\{a_1,a_2\}\!\}^{r''},e\right\}\!\!\right\}_\omega\\ &= \{\!\{a_1,a_2\}\!\}^{r'}\otimes\mathbb{D}_{\{\!\{a_1,a_2\}\!\}^{r''}}(e)\end{aligned}\tag{5.6.7}$$

and, similarly,

$$-\{\!\{a_2,a_1\}\!\}^{m''}\otimes\mathbb{T}_{\{\!\{a_2,a_1\}\!\}^{m'}}(e)=\{\!\{a_1,a_2\}\!\}^{m'}\otimes\mathbb{T}_{\{\!\{a_1,a_2\}\!\}^{m''}}(e)\tag{5.6.8}$$

Thus, by (5.6.6), (5.6.7) and (5.6.8),

(5.6.9)
$$\begin{aligned}\{\!\{\mathbb{D}_{a_1},\mathbb{D}_{a_2}\}\!\}_r &= -\{\!\{\mathbb{D}_{a_2},\mathbb{D}_{a_1}\}\!\}_l^{\circ}\\ &= -\sigma_{(12)}\left(\mathbb{D}_{\{\!\{a_2,a_1\}\!\}^{l'}}(e)\otimes\{\!\{a_2,a_1\}\!\}^{l''}+\mathbb{T}_{\{\!\{a_2,a_1\}\!\}^{m'}}(e)\otimes\{\!\{a_2,a_1\}\!\}^{m''}\right)\\ &= -\left(\{\!\{a_2,a_1\}\!\}^{l''}\otimes\mathbb{D}_{\{\!\{a_2,a_1\}\!\}^{l'}}(e)+\{\!\{a_2,a_1\}\!\}^{m''}\otimes\mathbb{T}_{\{\!\{a_2,a_1\}\!\}^{m'}}(e)\right)\\ &= \{\!\{a_1,a_2\}\!\}^{r'}\otimes\mathbb{D}_{\{\!\{a_1,a_2\}\!\}^{r''}}(e)+\{\!\{a_1,a_2\}\!\}^{m'}\otimes\mathbb{T}_{\{\!\{a_1,a_2\}\!\}^{m''}}(e)\end{aligned}$$

Finally, (5.6.4b) is the sum of (5.6.6) and (5.6.9), as required □

5.7 The isomorphism between A^2 and $\mathbb{A}\mathrm{t}(E_1)$

This subsection is devoted to prove that the map Ψ of twisted double Lie–Rinehart algebras (see Proposition 5.6.1) is an isomorphism, in the setting of double graded quivers considered in Proposition 3.3.34. In particular, this will imply the proof of the following non-commutative version of Roytenberg's result [81], Theorem 3.3:

Theorem 5.7.1. *Let (A,ω) be the pair consisting of the graded path algebra of a double quiver $\overline{P}$ of weight 2, and the bi-symplectic form $\omega\in\mathrm{DR}^2_R(A)$ of weight 2 defined in §3.3.4. Let B be the path algebra of the weight 0 subquiver of $\overline{P}$. Then (A,ω) is completely determined by the pair $(E_1,\langle-,-\rangle)$ consisting of the*

B-bimodule E_1 with basis consisting of weight 1 paths in P and the symmetric non-degenerate pairing

$$\langle -,-\rangle := \{\!\!\{-,-\}\!\!\}_\omega|_{E_1\otimes E_1} \to B\otimes B.$$

To prove that Ψ is an isomorphism, and hence Theorem 5.7.1, we will construct the following commutative diagram, where the rows are short exact sequences of B-bimodules.

(5.7.2)
$$\begin{tikzcd} 0 \arrow[r] & E_1\otimes_B E_1 \arrow[r] \arrow[d, "\Psi|_{E_1\otimes_B E_1}"] & A^2 \arrow[r] \arrow[d, "\Psi"] & E_2 \arrow[r] \arrow[d, "\widetilde{\imath}(\omega)_{(0)}"] & 0 \\ 0 \arrow[r] & \mathbb{ad}_{B^{\mathrm{e}}}(E_1) \arrow[r] & \mathbb{A}\mathrm{t}(E_1) \arrow[r] & \mathbb{D}\mathrm{er}_R\, B \arrow[r] & 0 \end{tikzcd}$$

Let

$$\mathbb{E}\mathrm{nd}_{B^{\mathrm{e}}}(E_1) := \mathrm{Hom}_{B^{\mathrm{e}}}(E_1, E_1\otimes B\oplus B\otimes E_1),$$

the *B-bimodule of double endomorphism of* E_1 and we define the *double adjoint B-bimodule of* $(E_1,\langle -,-\rangle)$ (see (5.3.9))

(5.7.3) $$\mathbb{ad}_{B^{\mathrm{e}}}(E_1) := \{\mathbb{D}\in \mathbb{E}\mathrm{nd}_{B^{\mathrm{e}}}(E_1)|-\langle e_1,\mathbb{D}_a(e_2)\rangle_L = \sigma_{(132)}\langle e_2,\mathbb{D}_a(e_1)^\circ\rangle_L\},$$

where $\langle -,-\rangle_L$ as in (5.3.8). Observe that the short exact sequences are given by the definitions of A^2 and $\mathbb{A}\mathrm{t}(E_1)$ (see (5.2.1) and Definition 5.5.4). Since $\widetilde{\imath}(\omega)_{(0)}$ is an isomorphism by Theorem 3.2.2, this implies that Ψ is an isomorphism if and only if so is its restriction to $E_1\otimes_B E_1$. The latter fact will be proved in the setting of graded double quivers developed in §3.3.4. From now on, let R be the semisimple commutative algebra with basis the trivial paths in P, hence B is a smooth R-algebra (see [59]).

The fact that the restriction of Ψ to $E_1\otimes_B E_1$ is an isomorphism will follow from the following tasks:

(i) Proof of an isomorphism $\mathbb{E}\mathrm{nd}_{B^{\mathrm{e}}}(E_1)\simeq (E_1\otimes_B E_1)\oplus(E_1\otimes_B E_1)$ (Lemma 5.7.4).

(ii) Description of a basis of $\mathbb{ad}_{\mathrm{e}}(E_1)$ (Proposition 5.7.11).

(iii) Description of $\Psi|_{E_1\otimes_B E_1}$ in the basis of (ii).

5.7.1 Description of the double endomorphisms

Lemma 5.7.4. *There is a canonical isomorphism*

$$\mathbb{E}\mathrm{nd}_{B^{\mathrm{e}}}(E_1)\simeq (E_1\otimes_B E_1)\oplus(E_1\otimes_B E_1).$$

Proof. By (3.3.25), $E_1 = \bigoplus_{|c|=1} BcB = \bigoplus_{|c|=1} Be_{h(c)} \otimes e_{t(c)}B$, so

$$E_1 \otimes_B E_1 \simeq \bigoplus_{|a|=|b|=1} Be_{h(b)} \otimes e_{t(b)}Be_{h(a)} \otimes e_{t(a)}B \simeq \bigoplus_{|a|=|b|=1} BbBaB. \tag{5.7.5}$$

(some over arrows a, b of weight 1), where we used that $B \otimes_B B \simeq B$. Now, the very description of E_1 just obtained enables us to describe in an explicit way the $(B^{\mathrm{e}})^{\mathrm{op}}$-module $\mathrm{Hom}_{B^{\mathrm{e}}}(E_1, B \otimes E_1)$:

(5.7.6)

$$\begin{aligned}
\mathrm{Hom}_{B^{\mathrm{e}}}(E_1, B \otimes E_1) \simeq & \bigoplus_{|a|=|b|=1} \mathrm{Hom}_{B^{\mathrm{e}}}\left(Be_{h(a)} \otimes e_{t(a)}B, B \otimes Be_{h(b)} \otimes e_{t(b)}B\right) \\
&= \bigoplus_{|a|=|b|=1} \mathrm{Hom}_B\left(Be_{h(a)}, B\right) \otimes \mathrm{Hom}_{B^{\mathrm{op}}}(e_{t(a)}B, Be_{h(b)} \otimes e_{t(b)}B\right) \\
&\simeq \bigoplus_{|a|=|b|=1} e_{h(a)}B \otimes Be_{h(b)} \otimes e_{t(b)}Be_{t(a)} \\
&\simeq \bigoplus_{|a|=|b|=1} Be_{h(b)} \otimes e_{t(b)}Be_{t(a)} \otimes e_{h(a)}B \\
&\simeq \bigoplus_{|a|=|b|=1} Be_{h(b)} \otimes e_{t(b)}Be_{h(a^*)} \otimes e_{t(a^*)}B \\
&\simeq \bigoplus_{|a|=|b|=1} BbBa^*B, \\
&\simeq \bigoplus_{|a|=|b|=1} BbBaB.
\end{aligned}$$

Now, $\overline{P}$ is a double graded quiver of weight 2, so there exists an isomorphism between the set of arrows a such that $|a| = 1$ and the set of reverse arrows $|a^*|$ which $|a^*| = 1$.

In the third identity in (5.7.6), we used that given a B-bimodule M and a B^{op}-bimodule N, there exist canonical isomorphisms (see (3.3.3))

$$\mathrm{Hom}_B(Be_i, M) \simeq e_iM, \quad \mathrm{Hom}_{B^{\mathrm{op}}}(e_iB, N) \simeq Ne_i. \tag{5.7.7}$$

In conclusion,

$$\mathrm{Hom}_{B^{\mathrm{e}}}(E_1, B \otimes E_1) \simeq \bigoplus_{|a|=|b|=1} BbBaB \simeq E_1 \otimes_B E_1.$$

In a similar way,

(5.7.8)

$$
\begin{aligned}
\operatorname{Hom}_{B^{\mathrm{e}}}(E_1, E_1\otimes B) &\simeq \bigoplus_{|a|=|b|=1} \operatorname{Hom}_{B^{\mathrm{e}}}\left(Be_{h(a)}\otimes e_{t(a)}B, Be_{h(b)}\otimes e_{t(b)}B\otimes B\right)\\
&\simeq \bigoplus_{|a|=|b|=1} \operatorname{Hom}_B(Be_{h(a)}, Be_{h(b)}\otimes e_{t(b)}B)\otimes \operatorname{Hom}_{B^{\mathrm{op}}}(e_{t(a)}B, B)\\
&\simeq \bigoplus_{|a|=|b|=1} (e_{h(a)}Be_{h(b)}\otimes e_{t(b)}B)\otimes Be_{t(a)}\\
&\simeq \bigoplus_{|a|=|b|=1} Be_{t(a)}\otimes e_{h(a)}Be_{h(b)}\otimes e_{t(b)}B\\
&\simeq \bigoplus_{|a|=|b|=1} Ba^*BbB,
\end{aligned}
$$

which also isomorphic to (5.7.5). □

To describe a basis of $\mathbb{a}\mathrm{d}_{B^{\mathrm{e}}}(E_1)$, we shall now write explicitly the element in $\operatorname{Hom}_{B^{\mathrm{e}}}({}_{B^{\mathrm{e}}}E_1, {}_BE_1\otimes B_B)$ which, under the isomorphisms in (5.7.8), corresponds to $ra^*qbp\in\bigoplus_{|a|=|b|=1}Ba^*BbB$, with r,q,p paths in B and the natural compatibility conditions

$$e_{h(p)}=e_{t(b)},\quad e_{h(b)}=e_{t(q)},\quad e_{h(q)}=e_{h(a)},\quad e_{t(a)}=e_{t(r)}.$$

Here we used our convention that paths compose from right to left. The obtained element will be denoted $[ra^*qbp]_1\in\operatorname{Hom}_{B^{\mathrm{e}}}({}_{B^{\mathrm{e}}}E_1, {}_BE_1\otimes B_B)$. If r',q',p' are paths in B, by a similar process, we determine $[r'bq'a^*p']_2\in\operatorname{Hom}_{B^{\mathrm{e}}}({}_{B^{\mathrm{e}}}E_1, {}_BB\otimes(E_1)_B)$ from $r'bq'a^*p'\in\bigoplus_{|a|=|b|=1}BbBa^*B$ by means of (5.7.6). By (5.7.7) and the canonical isomorphisms presented in §2.1.2, it is not difficult to see that:

(5.7.9)
$$
\begin{aligned}
(E_1\otimes_B E_1)\oplus(E_1\otimes_B E_1) &\xrightarrow{\cong} \mathbb{E}\mathrm{nd}_{B^{\mathrm{e}}}(E_1)\\
(ra^*qbp,0) &\longmapsto [ra^*qbp]_1\\
(0,r'bq'a^*p') &\longmapsto [r'bq'a^*p']_2,
\end{aligned}
$$

where

$$[ra^*qbp]_1\colon E_1\longrightarrow {}_BE_1\otimes B_B\colon sc\bar{s}\longmapsto \delta_{ac}(se_{h(a)}qbp)\otimes(re_{t(a)}\bar{s}),$$

and

$$[r'bq'a^*p']_2\colon E_1\longrightarrow {}_BB\otimes(E_1)_B\colon sc\bar{s}\longmapsto \delta_{ac}(se_{h(a)}p')\otimes(r'bq'e_{t(a)}\bar{s}).$$

Consequently, a basis of $\mathbb{E}\mathrm{nd}_{B^{\mathrm{e}}}(E_1)$ is given by $[ra^*qbp]_1$ and $[r'bq'a^*p']_2$.

5.7.2 Description of a basis of $\mathbb{a}\mathrm{d}_{B^{\mathrm{e}}}(E_1)$

To obtain a basis of $\mathbb{a}\mathrm{d}_{B^{\mathrm{e}}}(E_1)$, we impose the condition

(5.7.10)
$$\langle a, f(b)\rangle_L = -\sigma_{(132)}\langle b, f(a)^\circ\rangle_L,$$

to a linear combination of our basis of $\mathbb{E}\mathrm{nd}_{B^{\mathrm{e}}}(E_1)$, where $\sigma_{(123)}\colon B\otimes B\otimes B\to B\otimes B\otimes B\colon b_1\otimes b_2\otimes b_3\mapsto b_2\otimes b_3\otimes b_1$.

Lemma 5.7.11. *A basis of* $\mathbb{ad}_{B^e}(E_1)$ *consists of the elements*

$$\varepsilon(b)[ra^*qbp]_1 - \varepsilon(a)[ra^*qbp]_2, \tag{5.7.12}$$

where p, q, r are paths in B and a, b arrows of weight 1, which satisfy the following compatibility conditions:

$$e_{h(p)} = e_{t(b)}, \quad e_{h(b)} = e_{t(q)}, \quad e_{h(q)} = e_{h(a)}, \quad e_{t(a)} = e_{t(r)}.$$

Proof. To prove (5.7.10), we write explicitly $f(b)$ and $f(a)^\circ$. Using (5.7.9), for all $f \in \mathbb{E}\mathrm{nd}_{B^e}(E_1)$,

$$f := \sum_{|c|=|d|=1} \alpha_{rc^*qdp}[rc^*qdp]_1 + \alpha'_{rc^*qdp}[rc^*qdp]_2,$$

so for all weight 1 arrows b,

$$\begin{aligned} f(b) &= \sum_{|c|=|d|=1} \big(\alpha_{rc^*qdp}(\delta_{cb}(e_{h(b)}qdp) \otimes (re_{t(b)})) + \alpha'_{rc^*qdp}(\delta_{d^*b}(e_{h()}p) \otimes (rdqe_{t(b)}))\big) \\ &= \sum_{|d|=1} \big((\alpha_{rb^*qdp}e_{h(b)}qdp) \otimes (re_{t(b)}) + (\alpha'_{rdqb^*p}e_{h(b)}p) \otimes (rdqe_{t(b)})\big) \end{aligned}$$

Next, we can compute the left hand side of (5.7.10) (using (5.3.6) and (5.3.8)):

$$\begin{aligned} \langle a, f(b)\rangle_L &= \langle a, \sum_{|d|=1} \big((\alpha_{rb^*qdp}e_{h(b)}qdp) \otimes (re_{t(b)}) + (\alpha'_{rdqb^*p}e_{h(b)}p) \otimes (rdqe_{t(b)})\big)\rangle_L \\ &= \sum_{|d|=1} \langle a, \alpha_{rb^*qdp}e_{h(b)}qdp\rangle \otimes (re_{t(b)}) \\ &= \sum_{|d|=1} \alpha_{rb^*qdp}e_{h(b)}q\langle a, d\rangle p \otimes (re_{t(b)}) \\ &= \varepsilon(a)(\alpha_{rb^*qa^*p}e_{h(b)}qe_{t(a)}) \otimes (e_{h(a)}p) \otimes (re_{t(b)}) \end{aligned}$$

Similarly, using the maps

$$[rc^*qdp]_1^\circ \colon E_1 \longrightarrow B_B \otimes {}_BE_1 \colon shs' \longmapsto \delta_{ch}(re_{t(c)}s') \otimes (se_{h(c)}qdp)$$

and

$$[rdqc^*p]_2^\circ \colon E_1 \longrightarrow (E_1)_B \otimes {}_BB \colon shs' \longmapsto \delta_{ch}(rdqe_{t(c)}s') \otimes (se_{h(c)}p)$$

and the fact that $(-)^\circ$ is linear, it is straightforward to compute $-\sigma_{(132)}\langle b, f(a)^\circ\rangle_L$ and the result follows. □

Observe that the above descriptions of $\mathbb{ad}_{B^e}(E_1)$ and $E_1 \otimes_B E_1$ given in Lemma 5.7.11 and (5.7.5) provide an isomorphism between these B -bimodules:

$$\begin{aligned} E_1 \otimes_B E_1 &\longrightarrow \mathbb{ad}_{B^e}(E_1) \\ ra^*qbp &\longmapsto \varepsilon(b)[ra^*qbp]_1 - \varepsilon(a)[ra^*qbp]_2. \end{aligned} \tag{5.7.13}$$

5.7.3 The isomorphism $\Psi|_{E_1\otimes_B E_1}$

In this subsection, we will compute the value of $\Psi|_{E_1\otimes_B E_1}$ at basis elements ra^*qbp of $E_1 \otimes_B E_1$:

Lemma 5.7.14. *The map Ψ restricts to an isomorphism $\Psi\colon E_1 \otimes_B E_1 \xrightarrow{\cong} \mathbb{ad}_{B^e}(E_1)$, given by*

$$\Psi(ra^*qbp) = [ra^*qbp]_1 - [ra^*qbp]_2,$$

where p, q, r are paths in B and a, b arrows of weight 1, which satisfy the following compatibility conditions:

$$e_{h(p)} = e_{t(b)}, \quad e_{h(b)} = e_{t(q)}, \quad e_{h(q)} = e_{h(a)}, \quad e_{t(a)} = e_{t(r)}.$$

Proof. Firstly, note that since $\{\!\{-,-\}\!\}_\omega$ is a double Poisson bracket of weight -2, $\mathbb{X}_{ra^*qbp}(b') = 0$ for all $b' \in B$, because by a simple application of the Leibniz rule and (5.3.2):

$$\mathbb{X}_{ra^*qbp}(b') = -\sigma_{(12)}\left(ra^*\left\{\!\!\left\{b', qbp\right\}\!\!\right\}_\omega + \left\{\!\!\left\{b', ra^*\right\}\!\!\right\}_\omega qbp\right) = 0.$$

Thus,

$$\Psi|_{E_1\otimes_B E_1}\colon E_1 \otimes_B E_1 \longrightarrow \mathbb{ad}_{B^e}(E_1). \tag{5.7.15}$$

We apply the (graded) Leibniz rule when c is an arrow of weight 1:

$$\begin{aligned} \mathbb{D}_{ra^*qbp}(c) &= \{\!\{ra^*qbp, c\}\!\}_\omega \\ &= -\sigma_{(12)}\{\!\{c, ra^*qbp\}\!\}_\omega \\ &= -\sigma_{(12)}\left(ra^*q\{\!\{c,b\}\!\}_\omega p + r\{\!\{c,a^*\}\!\}_\omega qbp\right). \end{aligned} \tag{5.7.16}$$

To compute $\{\!\{c,b\}\!\}_\omega$ and $\{\!\{c,a^*\}\!\}_\omega$ we shall use Proposition 2.3.18 and, as a consequence, to address the question of the description of the differential double Poisson bracket P in the setting of quivers (recall Definition 2.3.21). M. Van den Bergh shows in [96] §6.2, that the double Schouten–Nijenhuis bracket acquires a very simple form in this case:

Proposition 5.7.17 ([96], Proposition 6.2.1)**.** *Let $A = kQ$ and $a, b \in Q$. Then*

$$\begin{aligned} \{\!\{a,b\}\!\} &= 0 \\ \left\{\!\!\left\{\frac{\partial}{\partial a}, b\right\}\!\!\right\} &= \begin{cases} e_{h(a)} \otimes e_{t(a)} & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \\ \left\{\!\!\left\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}\right\}\!\!\right\} &= 0 \end{aligned}$$

Note that in our convention, arrows compose from to left. The following result provides an explicit description of the *differential double Poisson bracket* $P \in (\mathrm{T}_A\,\mathbb{D}\mathrm{er}_B\,A)_2$ (see §2.3.3) in the context of double graded quivers:

Proposition 5.7.18 ([96], Theorem 6.3.1)**.** *Let $A = k\overline{P}$ be the graded path algebra of the double graded quiver $\overline{P}$. Then A has the following differential double Poisson bracket*

$$P = \sum_{a \in \overline{P}} \frac{\partial}{\partial a} \varepsilon(a) \frac{\partial}{\partial a^*}. \tag{5.7.19}$$

Applying now Propositions 2.3.18, 5.7.18 and 5.7.17 we can calculate

$$\begin{aligned} \{\!\{c, b\}\!\}_\omega &= -\{\!\{c, \{P, b\}\}\!\}_L \\ &= \left\{\!\!\left\{c, m \circ \left(\varepsilon(b)(e_{h(b)} \otimes e_{t(b)}) * \frac{\partial}{\partial b^*}\right)\right\}\!\!\right\}_L \\ &= \sigma_{(12)} \left\{\!\!\left\{\varepsilon(b) \frac{\partial}{\partial b^*}, c\right\}\!\!\right\} \\ &= \varepsilon(b) e_{h(b)} \otimes e_{t(b)}, \end{aligned} \tag{5.7.20}$$

where $m\colon A \otimes A \to A \otimes A\colon a \otimes b \mapsto ab$ is the multiplication. Replacing b by a^* in (5.7.20), we obtain

$$\{\!\{c, a^*\}\!\}_\omega = -\varepsilon(a) e_{t(a)} \otimes e_{h(a)}. \tag{5.7.21}$$

Using now (5.7.20) and (5.7.21) in (5.7.16), we can conclude

$$\begin{aligned} \mathbb{D}_{ra^*qbp}(c) &= -\sigma_{(12)} \left(ra^*q \{\!\{c, b\}\!\}_\omega p + r \{\!\{c, a^*\}\!\}_\omega qbp\right) \\ &= \varepsilon(b) e_{h(a)} qbp \otimes re_{t(a)} - \varepsilon(a) e_{h(a)} p \otimes rbqe_{t(a)} \end{aligned}$$

So

$$\Psi|_{E_1 \otimes_B E_1}(ra^*qbp) = \varepsilon(b)[ra^*qbp]_1 - \varepsilon(a)[ra^*qbp]_2. \qquad \square$$

Chapter 6

Non-commutative Courant algebroids

In Chapter 6, we calculate the non-commutative structures that arise when we equip a graded bi-symplectic tensor algebra (A, ω) of weight 2 with a homological double derivation Q. Here, a double derivation Q on A is homological if it satisfies the "double Maurer–Cartan" equation $\{\!\{Q, Q\}\!\} = 0$, where $\{\!\{-, -\}\!\}$ is the double Schouten–Nijenhuis commutator. Since our calculations will be based on results of Chapter 4, we focus on the case where (A, ω) is a bi-symplectic graded path algebra of a double graded quiver (see (3.3.4)). The new algebraic structures will be called "double Courant–Dorfman algebras". They are non-commutative versions of the Courant–Dorfman algebras introduced by Roytenberg [83], that themselves are to Courant algebroids what Lie–Rinehart algebras are to Lie algebroids.

In §6.1, we start with a short review of the role of Courant algebroids in geometry and physics (§6.1.1) and their definition §6.1.2. In §6.2 we provide an algebraic reformulation of Roytenberg's correspondence between symplectic $\mathbb{N}Q$-manifolds of weight 2 and Courant algebroids. Finally, in §6.3.1 we define the central object of this chapter –double Courant–Dorfman algebras–, and show that a bi-symplectic $\mathbb{N}Q$-algebra (A, ω) attached to a double graded quiver $\overline{P}$ determines a double Courant–Dorfman algebra over the path algebra of the weight 0 subquiver Q of P.

6.1 Courant algebroids

6.1.1 A brief historical account of Courant algebroids

Following [62], the origins of Courant algebroids can be found in the article [25] written by A. Weinstein and T. Courant who interpreted Dirac's bracket appeared in [32] in a geometric way by constructing a framework where Poisson and pre-symplectic structures were unified. Two years later, in his thesis [26], Courant defined a skew-symmetric bracket on $\mathrm{T}\,M \oplus \mathrm{T}^*\,M$ (called the *Courant bracket*) which, in general, is not a Lie-algebra bracket because the Jacobi identity does not hold. For sections $X + \xi$ and $Y + \eta$ of $\mathrm{T}\,M \oplus \mathrm{T}^*\,M$, this bracket is

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}\,\mathrm{d}\,(i_X\eta - i_Y\xi).$$

Note that Dirac structures of Courant and Weinstein coincide with those of Irene Ya. Dorfman as defined in 1987 in her article [34].

In [70], Liu, Weinstein and Xu systematized the properties of the bracket introduced by Courant in the definition of *Courant algebroid*. This structure on a vector bundle $E \to M$ involves an antisymmetric bracket on the sections of E whose "Jacobi anomaly" has an explicit expression in terms of a bundle map $E \to \mathrm{T}\,M$ and a field of symmetric bilinear forms on E. In [80], D. Roytenberg twisted the bracket in a Courant algebroid by adding a symmetric term (already suggested in [70]). As a consequence, he sacrificed skew-symmetry, but he obtained an equivalent (and more natural) definition of a Courant algebroid because the Jacobi identity in this non skew-symmetric setting resembles a Leibniz rule.

Roytenberg plays a central role in this history since he had the insight of seeing Courant algebroids as graded objects. In [81], he proved that symplectic $\mathbb{N}Q$-manifolds of weight 2 are in one-to-one correspondence with Courant algebroids; this theorem will be explained in §6.2. Later, he defined algebraic relatives of Courant algebroids: Courant–Dorfman algebras [83].

N. Hitchin and M. Gualtieri developed *Generalized complex geometry* (see [49] and [45]) which can be regarded as a way of unifying complex and symplectic geometry by taking the idea that both structures should be thought as linear operations on $\mathrm{T} \oplus \mathrm{T}^*$.

Finally, Courant algebroids have turned out to be interesting objects in Physics. In [82] it was showed that the AKSZ procedure yields a canonical three-dimensional Topological Field Theory associated to any Courant algebroid. Moreover, in [88] (see also [89]), Ševera presented Courant algebroids as the natural framework to deal with two-dimensional variational problems and, as a consequence, they seem the natural framework of string theory.

6.1.2 Definition of Courant algebroids

From now on, k is a field of characteristic zero, R is a commutative algebra and B is a smooth commutative R-algebra.

As in §5.1, (A, ω) is a symplectic polynomial $\mathbb{N}$-algebra of weight 2 such that $A = \mathrm{Sym}_B(E_1[-1] \oplus E_2[-2])$, where E_1 and E_2 are projective finitely generated B-modules, and $\mathrm{Sym}_B(-)$ denotes the graded symmetric algebra over B. Then, by Lemma 4.1.12, this structure induces a Poisson bracket of weight -2 (which is denoted $\{-,-\}_\omega$) and it is completely determined by the projective finitely generated B-module E_1 equipped with a symmetric non-degenerate bilinear form (the *inner product*) $\langle -,-\rangle\colon E_1\times E_1\to B$ (see §5.1.6). In order to simplify our exposition, we make the identification $E := E_1$.

Definition 6.1.1. A *pre-Courant algebroid over* B is a 4-tuple $(E,\langle -,-\rangle,\rho,\llbracket -,-\rrbracket)$ consisting of is a projective finitely generated B-module E endowed with a symmetric non-degenerate bilinear form (the *inner product*),

$$\langle -,-\rangle\colon E\times E\longrightarrow B,$$

a B-module morphism $\rho\colon E\to \operatorname{Der}_R B$, called the *anchor*, and a R-bilinear operation

$$\llbracket -,-\rrbracket\colon E\times E\longrightarrow E,$$

called the *Dorfman bracket.* These data must satisfy the following conditions:

$$\llbracket e_1,be_2\rrbracket=\rho(e_1)(b)e_2+b\llbracket e_1,e_2\rrbracket, \tag{6.1.2a}$$

$$\llbracket e_1,e_1\rrbracket=\frac{1}{2}\rho^*\,\mathrm{d}(\langle(e_1,e_1)\rangle), \tag{6.1.2b}$$

$$\rho(e_1)(\langle e_2,e_2\rangle)=2\langle\llbracket e_1,e_2\rrbracket,e_2\rangle, \tag{6.1.2c}$$

for all $b\in B$ and $e_1,e_2\in E$. Moreover, if the bracket satisfies the Jacobi identity

$$\llbracket e_1,\llbracket e_2,e_3\rrbracket\rrbracket=\llbracket\llbracket e_1,e_2\rrbracket,e_3\rrbracket+\llbracket e_2,\llbracket e_1,e_3\rrbracket\rrbracket, \tag{6.1.3}$$

for all $e_1,e_2,e_3\in E$, then $(E,g,\rho,\llbracket -,-\rrbracket)$ is called a *Courant algebroid over* B.

In (6.1.2b), d is the de Rham differential, $E\simeq E^*$ via $\langle -,-\rangle$ and $\rho^*\colon\Omega^1_R A\to E^*\simeq E$ is the dual map of ρ. In other words, $\langle e,\rho^*\,\mathrm{d}b\rangle=\rho(e)(f)$ for all $b\in B$ and $e\in E$.

Remark 6.1.4. Observe that Definition 6.1.1 is an algebraic adaptation of [44], §5.1. We point out that this definition is equivalent to [83], Definition 2.1. Roytenberg defines a Courant–Dorfman algebra as a 5-tuple $(\mathcal{R},\mathcal{E},\langle -,-\rangle,\partial,[-,-])$. The difference with respect to Definition 6.1.1 is the derivation $\partial\colon B\to E$ is a derivation, but, by (6.1.2a), $\rho(e)b=\langle e,\partial b\rangle$ and it follows the equivalence of both definitions.

Remark 6.1.5. Following [81] (and [80]), in this thesis we shall use the notion of Courant algebroid which satisfies the Jacobi identity (6.1.3) but it is not skew-symmetric. By (6.1.2b), it is not difficult to see that the *Dorfman bracket* satisfies the following identity:

$$\llbracket e_1,e_2\rrbracket=-\llbracket e_2,e_1\rrbracket+\rho^*\,\mathrm{d}g(e_1,e_2)$$

Note that there exists an equivalent notion of Courant algebroid, in which the operation (called the *Courant bracket*) is skew-symmetric but it satisfies the Jacobi identity only *up to an exact term* (given in terms of the derivative of the *Jacobiator*) (see [45], §3.2 and [80], Definition 2.3.2).

6.1.3 Connections and torsion on Courant algebroids

Let E be a Courant algebroid over B, ρ its anchor and F another finitely generated projective B-module.

Definition 6.1.6 (E-connection, [3])**.** A linear map $\nabla\colon E\otimes F\to F$ is called an *E-connection* on F if it satisfies the following properties,

$$\nabla_{be}f = b\nabla_e f, \tag{6.1.7a}$$

$$\nabla_e(bf) = b\nabla_e f + \rho(e)(b)f, \tag{6.1.7b}$$

where $e\in E$, $f\in F$ and $b\in B$.

Note that given an ordinary connection ∇ over F, we can define an E-connection using $\nabla_e s := \nabla_{\rho(e)}s$.

One can find at least three definitions of torsion of an E-connection ([3] and [44], [46] and [54]). In this thesis, we will use a variant of [54], with a slight change in the sign conventions.

Definition 6.1.8 (E-torsion)**.** Let $(E, g, \rho, [\![-,-]\!])$ be a Courant algebroid over B with a connection ∇. Then the weight 3 map $\mathrm{C}_\nabla\colon (E^*)^{\otimes 3}\to B$ defined by

$$\langle \mathrm{C}_\nabla, e_1\otimes e_2\otimes e_3\rangle = \langle [\![e_1,e_2]\!] - \nabla_{\rho(e_1)}e_2 + \nabla_{\rho(e_2)}e_1, e_3\rangle + \langle \nabla_{\rho(e_3)}e_1, e_2\rangle, \tag{6.1.9}$$

will be called the *E-torsion*.

Observe that this map is linear in e_3.

6.2 Courant algebroids and symplectic $\mathbb{N}Q$-algebras of weight 2

Yvette Kosmann-Schwarzbach introduced the notion of *derived bracket* in [61] showing that the Dorfman bracket on $\mathrm{T}\oplus\mathrm{T}^*$ is a derived bracket of the commutator of endomorphisms of the space of differential forms by the de Rham differential. Moreover, Vaintrob in [94] interpreted Lie algebroids as an odd-self-commuting vector field on a supermanifold. In [81], Roytenberg proved the equivalence between Courant algebroids and symplectic $\mathbb{N}Q$-algebras of weight 2, which will be explained in this section. The idea is that the structure of a Courant algebroid is encoded by an element $S\in A^3$ such that $\{S,S\}_\omega = 0$. Observe that, in some sense, S generalizes the Cartan 3-form on a quadratic Lie algebra appearing in the Chern–Simons theory (see [82]).

6.2.1 Bijection between pre-Courant algebroids and weight 3 functions

Recall (see §4.1) that an $\mathbb{N}Q$-*manifold* $(\mathcal{M},Q)$ is an $\mathbb{N}$-manifold endowed with an integrable homological vector field Q of weight $+1$ (i.e. $[Q,Q]=2Q^2=0$, where

$[-,-]$ denotes the Schouten bracket of multi-derivations). In addition, a *symplectic* $\mathbb{N}Q$*-manifold* $(\mathcal{M},\omega,Q)$ is an $\mathbb{N}Q$-manifold whose homological vector field is compatible with ω in the sense that $\mathrm{L}_Q\omega=0$, where L_Q stands for the Lie derivative along the vector field Q.

Let R be a commutative k-algebra, where k is a field of characteristic zero and B a smooth commutative R-algebra. Following §4.1.3 (and [20]), if (A,ω,Q) is a symplectic $\mathbb{N}Q$-algebra over B of weight 2 (see Definition 4.1.15) with $A^0=B$, we shall show that the relation $[Q,Q]=0$ is equivalent to $\{S,S\}_\omega=0$, where $S\in A^3$ and $\{-,-\}_\omega$ is the Poisson bracket of weight -2 induced by ω via (4.1.7).

Since Q is a symplectic homological derivation, it satisfies $\mathrm{L}_Q\omega=0$ and $[Q,Q]=0$, by Lemma 4.1.6(ii), we have

$$Q=H_S=\{S,-\}_\omega,$$

where $S\in A$. In fact, by definition, Q has weight $+1$ and the identity for all $b\in B$ $\{S,b\}_\omega=H_S(b)=Q(b)$ yields

$$|S|=|Q|-|\{-,-\}_\omega|=3.$$

Thus $S\in A^3$. Now, by the Jacobi identity we have the identity $H_{\{a,b\}_\omega}=[H_a,H_b]$ which applied to $a=b=S$ gives

$$H_{\{S,S\}_\omega}=[Q,Q].$$

Therefore, if $a\in A$,

$$[Q,Q](a)=H_{\{S,S\}_\omega}(a)=\{\{S,S\}_\omega,a\}_\omega,$$

and the relation $[Q,Q]=0$ is equivalent to $\{S,S\}_\omega\in B$ because the equation (4.1.4) provides $\{S,S\}_\omega\in R$ but B is a commutative R-algebra. However, $\{S,S\}_\omega$ has weight 4 because $|S|=3$. Therefore, we conclude that $\{S,S\}_\omega=0$ as we claimed.

The following result is important because states that the function $S\in A^3$ obtained in the previous discussion encodes the structure of pre-Courant algebroid introduced in Definition 6.1.1:

Proposition 6.2.1. *Every weight 3 function $S\in A^3$ induces a pre-Courant algebroid structure on (E,g) by setting*

$$\rho(e_1)(b):=\{\{S,e_1\}_\omega,b\}_\omega, \tag{6.2.2a}$$

$$[\![e_1,e_2]\!]:=\{\{S,e_1\}_\omega,e_2\}_\omega, \tag{6.2.2b}$$

for all $b\in B$ and $e_1,e_2\in E$. Conversely, given a pre-Courant algebroid, we can construct the weight 3 function $S=\rho+C_\nabla$ which satisfies (6.2.2).

Proof. The Poisson bracket of weight -2 $\{-,-\}_\omega$ induced by the symplectic form ω will be denoted by $\{-,-\}$ whereas the inner product $\langle -,-\rangle$ by $g(-,-)$. By Definition 6.1.1, we have to check (6.1.2a) - (6.1.2c). Using the fact that $\{-,-\}$ is a derivation in its second argument,

$$\begin{aligned}[\![e_1, be_2]\!] &= b\{\{S,e_1\},e_2\} + \{\{S,e_1\},b\}e_2 \\ &= b[\![e_1,e_2]\!] + \rho(e_1)(b)e_2.\end{aligned}$$

To prove (6.1.2b), since $\{-,-\}$ is a bracket of weight -2, $\{e_1,\{e_2,e_2\}\} = 0$ when $e_1, e_2 \in E$. Then, applying the Jacobi identity (4.1.11):

$$\begin{aligned}0 &= \{S,\{e_1,\{e_2,e_2\}\}\} \\ &= \{\{S,e_1\},\{e_2,e_2\}\} - \{e_1,\{S,\{e_2,e_2\}\}\} \\ &= \{\{S,e_1\},\{e_2,e_2\}\} - \{e_1,\{\{S,e_2\},e_2\} - \{e_2,\{S,e_2\}\}\} \\ &= \{\{S,e_1\},\{e_2,e_2\}\} - 2\{e_1,\{\{S,e_2\},e_2\}\}\},\end{aligned}$$

where we used that $\{e_2,\{S,e_2\}\} = -\{\{S,e_2\},e_2\}$. Hence, since $\{\xi,\eta\} = g(\xi,\eta)$, when $\xi,\eta \in A^1$, we obtain identity

$$\rho(e_1)(g(e_2,e_2)) = 2g(e_1, [\![e_2,e_2]\!]).$$

Finally, as $\rho(e_1)(g(e_2,e_2)) = g(e_1, \rho^* \, \mathrm{d}g(e_2,e_2))$ and g is a non-degenerate bilinear form by hypothesis, we get (6.1.2b). The proof of (6.1.2c) is a simple application of the Jacobi identity:

$$\begin{aligned}\rho(e_1)(g(e_2,e_2)) &= \{\{S,e_1\},\{e_2,e_2\}\} \\ &= \{\{\{S,e_1\},e_2\},e_2\} + \{e_2,\{\{S,e_1\},e_2\}\} \\ &= g([\![e_1,e_2]\!],e_2) + g([\![e_1,e_2]\!],e_2) \\ &= 2g([\![e_1,e_2]\!],e_2),\end{aligned}$$

where, again, we used that $\{e_1,e_2\} := g(e_1,e_2)$ for $e_1,e_2 \in E$.

Conversely, given a pre-Courant algebroid, we construct the weight 3 function $S = \rho + C_\nabla$ and we shall prove that it satisfies (6.2.2). *This proof was kindly communicated to the author by Jean-Philippe Michel.*

First observe that both the torsion map and the anchor map ρ are weight 3 functions; ρ can be identified with an element of $E \otimes \operatorname{Der}_R B$ using the inner product g (recall that $E_2 \simeq \operatorname{Der}_R B$). Thus, $S = \rho + \mathrm{C}_\nabla$ is a weight 3 function on A.

Next, it is easy to see that S satisfies (6.2.2a). To prove that S satisfies (6.2.2b), using sheaf theory, recall that a *parallel section* $e \in E$ satisfies $\nabla_X e = 0$ for all $X \in \operatorname{Der}_R B$. Furthermore, a general section is a finite linear combination of terms of the form be, where $b \in B$ and $e \in E$ is a parallel section. Now, we shall write the structure of the proof:

(i) The equation

$$\{\{S, e_1\}, e_2 = [\![e_1, e_2]\!] \tag{6.2.3}$$

holds for any parallel sections $e_1, e_2 \in E$.

(ii) The equation

$$\{\{S, e_1\}, fe_2\} = [\![e_1, be_2]\!]$$

holds for any parallel sections $e_1, e_2 \in E$ and $b \in B$. In other words, (6.2.3) holds for any parallel section e_1 and any section e_2.

(iii) (6.2.3) holds for any section e_1 and any parallel section e_2.

(iv) (6.2.3) holds for any sections e_1 and e_2.

In (i), for any e_1, e_2, e_3, we have

$$\{\{\{\mathrm{C}_\nabla, e_1\}, e_2\}, e_3\} = \mathrm{C}_\nabla(e_1, e_2, e_3), \quad \{\{\{\rho, e_1\}, e_2\}, e_3\} = g(\nabla_{\rho(e_1)} e_2, e_3),$$

where we used that $\{e, e'\} = g(e, e')$ for all $e, e' \in E$. Hence,

$$\{\{\{S, e_1\}, e_2\}, e_3\} = \mathrm{C}_\nabla(e_1, e_2, e_3) + g(\nabla_{\rho(e_1)} e_2, e_3),$$

which is B-linear in e_3. If we restrict it to parallel sections, we obtain the following expression (recall (6.1.9))

$$\{\{\{\mathrm{C}_\nabla, e_1\}, e_2\}, e_3\} = g([\![e_1, e_2]\!], e_3), \quad \{\{\{\rho, e_1\}, e_2\}, e_3\} = 0.$$

Thus, for parallel sections,

$$\{\{\{S, e_1\}, e_2\}, e_3\} = g([\![e_1, e_2]\!], e_3),$$

Both sides of this equation are B-linear in e_3, so it holds for any e_3. Since g is non-degenerate, by hypothesis, we conclude that (6.2.3) holds in the case when e_1 and e_2 are parallel sections.

Next, in (ii), consider parallel sections e_1 and e_2 and $b \in B$. By (i) and the Leibniz rule,

$$\{\{S, e_1\}, be_2\} = \rho(e_1)(b)e_2 + b[\![e_1, e_2]\!] = [\![e_1, be_2]\!],$$

where in the last equation we used (6.1.2a). Therefore, (6.2.3) is true for any parallel section e_1 and any section e_2.

In (iii), consider sections e_1 and e_2. Then, polarizing (6.1.2b), we obtain that $[\![e_1, e_2]\!] = [\![e_2, e_1]\!] + \rho^* \, \mathrm{d}g(e_1, e_2)$. By the Jacobi identity,

$$\begin{aligned}\{\{S, e_1\}, e_2\} &= -\{\{S, e_1\}, e_2\} + \{S, \{e_1, e_2\}\} \\ &= -\{\{S, e_1\}, e_2\} + \rho^* \, \mathrm{d}g(e_1, e_2).\end{aligned}$$

So, if (6.2.3) holds for $[\![e_1, e_2]\!]$, then it holds also for $[\![e_2, e_1]\!]$. Together with (ii), this implies that (6.2.3) holds for any section e_1, and any parallel section e_2.

Finally, in the last item, consider parallel sections e_1 and e_2 and $b, b' \in B$. Then, applying (6.1.2a),

$$\begin{aligned}
[\![be_1, b'e_2]\!] &= b' [\![be_1, e_2]\!] + \rho(be_1)(b')e_2 \\
&= b'\{\{S, be_1\}, e_2\} + \{\{S, be_1\}, b'\}e_2 \\
&= \{\{S, be_1\}, b'e_2\},
\end{aligned}$$

where in the second equation we applied (iii) and (6.2.2a), whereas in the last one, we used the Leibniz rule for $\{-,-\}$. So, we can conclude that (6.2.3) holds for any sections e_1 and e_2. $\square$

6.2.2 Courant algebroids and the homological condition

Proposition 6.2.4. *The Jacobi identity* (6.1.3) *for* $[\![-,-]\!]$ *is equivalent to the homological condition* $\{S, S\}_\omega = 0$.

Proof. The Poisson bracket of weight -2 $\{-,-\}_\omega$ induced by the symplectic form ω will be denoted by $\{-,-\}$ whereas the inner product $\langle -,- \rangle$ by $g(-,-)$. We shall prove the following Jacobi identity

$$[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]. \tag{6.2.5}$$

(6.2.2b) provides

$$\begin{aligned}
[\![e_1, [\![e_2, e_3]\!]]\!] &= \{\{S, e_1\}, [\![e_2, e_3]\!] \} \\
&= \{\{S, e_1\}, \{\{S, e_2\}, e_3\}\} \\
&= \{\{\{S, e_1\}, \{S, e_2\}\}, e_3\} + \{\{S, e_2\}, \{\{S, e_1\}, e_3\}\},
\end{aligned}$$

where we applied the Jacobi identity (4.1.11) for $\{-,-\}$. By the same reason and using that $\{S, S\} = 0$,

$$\begin{aligned}
\{S, \{\{S, e_1\}, e_2\}\} &= \{\{S, \{S, e_1\}\}, e_2\} + \{\{S, e_1\}, \{S, e_2\}\} \\
&= \{\frac{1}{2}\{\{S, S\}, e_1\} + \{\{S, e_1\}, \{S, e_2\}\} \\
&= \{\{S, e_1\}, \{S, e_2\}\},
\end{aligned}$$

Then,

$$\begin{aligned}
[\![e_1, [\![e_2, e_3]\!]]\!] &= \{S, \{\{S, e_1\}, e_2\}\} + \{\{S, e_2\}, \{\{S, e_1\}, e_3\}\} \\
&= [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!].
\end{aligned}$$

Conversely, following [54], it is not difficult to see that as a consequence of the Jacobi identity for $[\![-,-]\!]$,we obtain the expression for all $e_1, e_2, e_3 \in E$.

$$\{\{\{\{S, S\}, e_1\}, e_2\}, e_3\} = 0 \tag{6.2.6}$$

Now, let $b \in B$. Then, by the Leibniz rule for $\{-,-\}$ in the second argument,

$$\begin{aligned} 0 &= \{\{\{\{S,S\},e_1\},e_2\},be_3\} \\ &= b\{\{\{\{S,S\},e_1\},e_2\},e_3\} + \{\{\{\{S,S\},e_1\},e_2\},b\}e_3. \end{aligned}$$

By (6.2.6), we obtain

$$\{\{\{\{S,S\},e_1\},e_2\},b\} = 0 \tag{6.2.7}$$

Finally, by another application of the Jacobi identity and (6.2.7), for all $e_1, e_2 \in E$ and $b \in B$,

$$\begin{aligned} 0 &= \{\{\{\{S,S\},e_1\},e_2\},b\} \\ &= \{\{\{S,S\},e_2\},e_1\},b\} + \{\{\{S,S\},\{e_1,e_2\}\},b\} \\ &= \{\{\{S,S\},\{e_1,e_2\}\},b\} \end{aligned}$$

The key point now is that every $b' \in B$ can be written $b' = \{e_1,e_2\}$ with appropriate $e_1,e_2 \in E$. Therefore, we conclude that

$$\{\{\{S,S\},b'\},b\} = 0,$$

for all $b, b' \in B$, which implies that $\{S,S\} = 0$, as required. □

6.3 Non-commutative Courant algebroids

6.3.1 Definition of double Courant–Dorfman algebras

Following Definition 6.1.1, in this section we introduce the notion of a *double Courant–Dorfman algebras.*

Let k be a field of characteristic zero, R be a finite dimensional semisimple associative k-algebra and B an associative R-algebra. Recall that in Chapter 3 and Chapter 5, we proved that the structure of a given bi-symplectic tensor $\mathbb{N}$-algebra (A,ω) of weight 2 corresponds to a pair $(E_1, \langle -,-\rangle)$, where E_1 is a projective finitely generated B-bimodule and $\langle -,-\rangle$ is the symmetric non-degenerate pairing defined in (5.3.6) (see Theorem 5.7.1). To simplify our exposition, we make the identification $E := E_1$ and, if necessary, $\langle -,-\rangle := \langle -,-\rangle_L$, which was defined in (5.3.8).

Definition 6.3.1 (Double Courant–Dorfman algebra)**.** A *double pre-Courant–Dorfman algebra over* B is a 4-tuple $(E, \langle -,-\rangle, \rho, [\![-,-]\!])$ consisting of a projective finitely generated B-bimodule E endowed with a symmetric non-degenerate pairing (the *inner product*)

$$\langle -,-\rangle \colon E \otimes E \longrightarrow B \otimes B,$$

a B-bimodule morphism

$$\rho \colon E \longrightarrow \mathbb{D}\mathrm{er}_R\, B, \tag{6.3.2}$$

called the *anchor*, and an operation

$$(6.3.3)\qquad [\![-,-]\!]: E\otimes E \longrightarrow (E\otimes B)\oplus(B\otimes E),$$

called the *double Dorfman bracket*, which is R-linear for the outer bimodule structure on $B\otimes B$ in the second argument and R-linear for the inner bimodule structure on $B\otimes B$ in the first argument. These data must satisfy the following conditions:

$$(6.3.4a)\qquad [\![e_1, be_2]\!] = \rho(e_1)(b)e_2 + b[\![e_1,e_2]\!],$$
$$(6.3.4b)\qquad [\![e_1, e_2 b]\!] = e_2\rho(e_1)(b) + [\![e_1,e_2]\!]b,$$
$$(6.3.4c)\qquad [\![be_1, e_2]\!] = e_1 * \sigma_{(12)}\rho(e_2)(b)e_1 + b * [\![e_1,e_2]\!],$$
$$(6.3.4d)\qquad [\![e_1 b, e_2]\!] = e_1 * \sigma_{(12)}\rho(e_2)(b) + [\![e_1,e_2]\!] * b,$$
$$(6.3.4e)\qquad \rho^\vee \mathrm{d}\langle e_1,e_1\rangle = [\![e_1,e_1]\!] + [\![e_1,e_1]\!]^\circ,$$
$$(6.3.4f)\qquad \rho(e_1)(\langle\!\langle e_2,e_2\rangle\!\rangle) = \tau_{(132)}\langle e_2, [\![e_1,e_2]\!]\rangle + \tau_{(123)}\langle e_2, [\![e_1,e_2]\!]^\circ\rangle,$$

for all $b\in B$ and $e_1,e_2\in E$. If the bracket $[\![-,-]\!]$ satisfies the *double Jacobi identity*:

$$(6.3.5)\qquad \tau_{(123)}[\![e_1,[\![e_2,e_3]\!]]\!]_L + \tau_{(123)}[\![e_2,[\![e_1,e_3]\!]^\circ]\!]_L + [\![[\![e_1,e_2]\!],e_3]\!]^\circ_L = 0$$

for all $e_1,e_2,e_3\in E$, then $(E,\langle -,-\rangle,\rho,[\![-,-]\!])$ is called a *double Courant–Dorfman algebra*.

In (6.3.4c) and (6.3.4d), $\sigma_{(12)}$ denotes the permutation $A\otimes A\to A\otimes A$: $a_1\otimes a_2\mapsto a_2\otimes a_1$, whereas in (6.3.4e) $\mathrm{d}\colon A\to\Omega^1_R A$ is the de Rham differential, and $\rho^\vee\colon\Omega^1_R A\to E^\vee\simeq E$ is the composite of the canonical map $\Omega^1_R B\to(\mathbb{D}\mathrm{er}_R\, B)^\vee$, $\mathrm{Hom}_{B^\mathrm{e}}(\rho, {}_{B^\mathrm{e}}B^\mathrm{e})\colon(\mathbb{D}\mathrm{er}_R\, B)^\vee\to E^\vee$, and the isomorphism $E^\vee\simeq E$ induced by $\langle -,-\rangle$ (see (5.3.5)). By convention, in (6.3.4e),

$$\begin{aligned}\rho^\vee\mathrm{d}\langle e_1,e_1\rangle &= \rho^\vee\mathrm{d}(\langle e_1,e_1\rangle'\otimes\langle e_1,e_1\rangle'')\\ &= \rho^\vee(\mathrm{d}(\langle e_1,e_1\rangle')\otimes\langle e_1,e_1\rangle'' + \langle e_1,e_1\rangle'\otimes\rho^\vee(\mathrm{d}(\langle e_1,e_1\rangle'')).\end{aligned}$$

Similarly, we use this convention for $\rho(e_1)(\langle\!\langle e_2,e_2\rangle\!\rangle)$ in (6.3.4f). As in the commutative case, we will use of the identification

$$\langle e,\rho^\vee\mathrm{d}b\rangle = \rho(e)(b)$$

for all $e\in E$ and $b\in B$.

It will be useful to define $\langle -,-\rangle\colon E\otimes B\oplus B\otimes E\to B\otimes B\otimes B$ by the formulae (see (5.3.8))

$$\langle e\otimes b, e'\rangle = 0,\qquad \langle b\otimes e, e'\rangle = b\otimes\langle e,e'\rangle.$$

It will also be useful to apply Sweedler's notation to the double Dorfman bracket:

$$[\![-,-]\!] = [\![-,-]\!]_l + [\![-,-]\!]_r,$$

where

$$\llbracket -,-\rrbracket_l = \llbracket -,-\rrbracket_l' \otimes \llbracket -,-\rrbracket_l'' \colon E\otimes E \longrightarrow E\otimes B,$$
$$\llbracket -,-\rrbracket_r = \llbracket -,-\rrbracket_r' \otimes \llbracket -,-\rrbracket_r'' \colon E\otimes E \longrightarrow B\otimes E,$$

Then, for $i=l,r$,

$$\llbracket e,e'\rrbracket_i^\circ = -\llbracket e',e\rrbracket_i'' \otimes \llbracket e',e\rrbracket_i'$$

and so the brackets appearing in the double Jacobi identity (6.3.5) are sums of terms of the following types

$$\begin{aligned}
\llbracket e, \llbracket e', e''\rrbracket_i' \otimes \llbracket e',e''\rrbracket_i''\rrbracket_L &= \llbracket e, \llbracket e',e''\rrbracket_i'\rrbracket \otimes \llbracket e',e''\rrbracket_i'',\\
\llbracket \llbracket e',e''\rrbracket_i' \otimes \llbracket e',e''\rrbracket_i'', e\rrbracket_L &= \llbracket \llbracket e',e''\rrbracket_i', e\rrbracket \otimes \llbracket e',e''\rrbracket_i'',\\
\llbracket e, \llbracket e',e''\rrbracket_i' \otimes \llbracket e',e''\rrbracket_i''\rrbracket_L^\circ &= \llbracket e, \llbracket e',e''\rrbracket_i'\rrbracket^\circ \otimes \llbracket e',e''\rrbracket_i'',\\
\llbracket \llbracket e',e''\rrbracket_i' \otimes \llbracket e',e''\rrbracket_i'', e\rrbracket_L^\circ &= \llbracket \llbracket e',e''\rrbracket_i', e\rrbracket^\circ \otimes \llbracket e',e''\rrbracket_i'',
\end{aligned}$$

for all $e,e',e''\in E$.

6.4 Double Courant algebroids and bi-symplectic $\mathbb{N}Q$-algebras of weight 2

The main goal of this section is to explore the connection between bi-symplectic $\mathbb{N}Q$-algebras of weight 2 (see Definition 4.3.1) and double Courant–Dorfman algebras. As in the commutative case, by Lemma 2.5.6, a bi-symplectic double derivation is a Hamiltonian double derivation. Thus, we can write $Q=\{\!\{S,-\}\!\}_\omega$, where $\{\!\{-,-\}\!\}_\omega$ is the double Poisson bracket of weight -2 induced by ω and $S\in A^3$. The idea is that we should expect such a S encodes the structure of a double pre-Courant–Dorfman algebra, recoverable via derived brackets and, if in addition, $\{S,S\}_\omega=0$ (here $\{-,-\}_\omega$ denotes the associated bracket to $\{\!\{-,-\}\!\}_\omega$), then we shall obtain the structure of a double Courant–Dorfman algebra.

From now on, we fix the following framework: let R be a finite dimensional semisimple associative k-algebra, where k is a field of characteristic zero, and B a smooth associative R-algebra. Let (A,ω,Q) be a bi-symplectic $\mathbb{N}Q$-algebra of weight 2 over R such that $A^0=B$. In particular, let Q be a bi-symplectic homological double derivation, that is, $|Q|=+1$, $\mathcal{L}_Q\omega=0$ and $\{\!\{Q,Q\}\!\}=0$, where $\{\!\{-,-\}\!\}$ is the graded double Schouten–Nijenhuis bracket on $\mathrm{T}_A\,\mathbb{D}\mathrm{er}_R A$. First, we show that the identity $\{\!\{Q,Q\}\!\}=0$ is equivalent to $\{\!\{S,S\}\!\}_\omega=0$.

By Lemma 2.5.6(ii), $\iota_Q\omega=\mathrm{d}S$ for some $\theta\in A$, that is,

$$Q=\{\!\{S,-\}\!\}_\omega \tag{6.4.1}$$

where $S\in A$. Now, for all $a\in A$, the identity $\{\!\{S,a\}\!\}_\omega=Q(a)$ implies

$$|S|=|Q|-|\{\!\{-,-\}\!\}_\omega|=3,$$

that is, $S \in A^3$. Now, the identity $H_{\{\!\{a,b\}\!\}_\omega} = \{\!\{H_a, H_b\}\!\}$ (see Proposition 2.4.6) applied to $a = b = S$ gives

$$H_{\{\!\{S,S\}\!\}_\omega} = \{\!\{Q, Q\}\!\} .$$

Therefore the relation $\{\!\{Q, Q\}\!\} = 0$ is equivalent to $\{\!\{S, S\}\!\}_\omega \in B \otimes B$ because, by (2.4.2), $H_{\{\!\{S,S\}\!\}_\omega} = 0$ implies that $0 = \mathrm{d}\{\!\{S, S\}\!\}_\omega = \mathrm{d}\{\!\{S, S\}\!\}'_\omega \otimes \{\!\{S, S\}\!\}''_\omega + \{\!\{S, S\}\!\}'_\omega \otimes \mathrm{d}\{\!\{S, S\}\!\}''_\omega$, which implies that $\{\!\{S, S\}\!\}'_\omega , \{\!\{S, S\}\!\}''_\omega \in R$. Finally, since B is an associative R-algebra, we conclude that $\{\!\{S, S\}\!\}'_\omega , \{\!\{S, S\}\!\}''_\omega \in B$ as we required and, as a consequence, $|\{\!\{S, S\}\!\}_\omega| = 0$. However, $\{\!\{S, S\}\!\}_\omega$ has weight 4 because $|S| = 3$. So $\{\!\{S, S\}\!\}_\omega = 0$.

Next, as we wrote above, we shall prove that the structure of double pre-Courant algebroid can be recovered by means of derived brackets as the following result shows:

Proposition 6.4.2. *Every weight 3 function $S \in A^3$ induces a double pre-Courant–Dorfman algebra structure on $(E, \langle -, -\rangle)$ by setting*

$$\rho(e_1)(b) := \{\!\{\{S, e_1\}_\omega, b\}\!\}_\omega , \tag{6.4.3a}$$

$$[\![e_1, e_2]\!] := \{\!\{\{S, e_1\}_\omega, e_2\}\!\}_\omega , \tag{6.4.3b}$$

for all $b \in B$ and $e_1, e_2 \in E$, and where $\{-, -\} = m \circ \{\!\{-, -\}\!\}_\omega$ is the associated bracket *in A (see (2.3.5)).*

Remark 6.4.4. In this section, we will use the associated bracket. Then we have a graded version of (2.3.8) (cf. [2])

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a||b|}\{b, \{a, c\}\}. \tag{6.4.5}$$

Finally, recall that $\{a, -\}$ acts on tensors by $\{a, u \otimes v\} := \{a, u\} \otimes v + u \otimes \{a, v\}$.

Proof. For sake of simplicity, the double Poisson bracket of weight -2 $\{\!\{-, -\}\!\}_\omega$ induced by the bi-symplectic form ω will be denoted by $\{\!\{-, -\}\!\}$ and the non-degenerate symmetric pairing $\langle -, -\rangle$ by $g(-, -)$. In order to prove (6.3.4a) in Definition 6.3.1, we shall use the fact that $\{\!\{-, -\}\!\}$ is a double derivation in its second argument with respect to the outer bimodule structure:

$$\begin{aligned}
[\![e_1, be_2]\!] &= \{\!\{\{S, e_1\}, be_2\}\!\} \\
&= b\,\{\!\{\{S, e_1\}, e_2\}\!\} + \{\!\{\{S, e_1\}, b\}\!\}\, e_2 \\
&= b\,[\![e_1, e_2]\!] + \rho(e_1)(b)e_2
\end{aligned}$$

The proof that $[\![e_1, e_2 b]\!] = e_2\rho(e_1)(b) + [\![e_1, e_2]\!]\, b$ holds is quite similar. (6.3.4c) and (6.3.4d) also follow easily. To prove (6.3.4e), note that since $\{\!\{-, -\}\!\}$ is a double Poisson bracket of weight -2, $\{\!\{e_1, \{\!\{e_2, e_2\}\!\}\}\!\}_L = 0$. Then, by (2.3.7):

$$\begin{aligned}
0 &= \{S, \{\!\{e_1, \{\!\{e_2, e_2\}\!\}\}\!\}_L\} \\
&= \{\!\{\{S, e_1\}, \{\!\{e_2, e_2\}\!\}\}\!\}_L - \{\!\{e_1, \{S, \{\!\{e_2, e_2\}\!\}\}\}\!\}_L \\
&= \{\!\{\{S, e_1\}, \{\!\{e_2, e_2\}\!\}\}\!\}_L - \{\!\{e_1, \{\!\{\{S, e_2\}, e_2\}\!\} - \{\!\{e_2, \{S, e_2\}\}\!\}\}\!\}_L \\
&= \{\!\{\{S, e_1\}, \{\!\{e_2, e_2\}\!\}\}\!\}_L - \{\!\{e_1, \{\!\{\{S, e_2\}, e_2\}\!\} + \{\!\{\{S, e_2\}, e_2\}\!\}^\circ\}\!\}_L
\end{aligned}$$

By definition, if $e, e' \in E$, $\{\!\{e, e'\}\!\} = g(e, e')$, so

$$\rho(e_1)(g(e_2, e_2)) = g(e_1, [\![e_2, e_2]\!] + [\![e_2, e_2]\!]^\circ)$$

Next, since g is non-degenerate, the identity

$$g(e_1, \rho^\vee \, \mathrm{d}g(e_2, e_2)) = g(e_1, [\![e_2, e_2]\!] + [\![e_2, e_2]\!]^\circ)$$

implies (6.3.4c). Finally, the key ingredient to prove (6.3.4d) holds is the double Jacobi identity for $\{\!\{-,-\}\!\}$ (see (2.3.12)):

$$\begin{aligned} -\rho(e_1)(g(e_2, e_2)) &= -\{\!\{\{S, e_1\}, g(e_2, e_2)\}\!\}_L \\ &= \tau_{(123)} \{\!\{e_2, \{\!\{e_2, \{S, e_1\}\}\!\}\}\!\}_L - \tau_{(132)} \{\!\{e_2, \{\!\{\{S, e_1\}, e_2\}\!\}\}\!\}_L \\ &= -\tau_{(123)} \{\!\{e_2, [\![e_1, e_2]\!]^\circ\}\!\}_L - \tau_{(132)} \{\!\{e_2, [\![e_1, e_2]\!]\}\!\}_L \\ &= -\tau_{(123)}\, g(e_2, [\![e_1, e_2]\!]^\circ) - \tau_{(132)}\, g(e_2, [\![e_1, e_2]\!]) \end{aligned}$$

□

Finally, at the beginning of this subsection, we showed that the fact that the homological double derivation Q satisfied $\{\!\{Q, Q\}\!\} = 0$ implied $\{\!\{S, S\}\!\}_\omega = 0$. In the next result, we prove that the weaker condition $\{S, S\}_\omega = 0$ (where $\{-,-\}_\omega$ is the associated bracket) implies the double Jacobi identity (6.3.5).

Proposition 6.4.6. *If $\{S, S\}_\omega = 0$ then the double Jacobi identity in (6.3.5) holds.*

Proof. As usual, $\{\!\{-,-\}\!\}_\omega$ to be denoted $\{\!\{-,-\}\!\}_\omega$. By (6.4.3b),

$$[\![e_1, [\![e_2, e_3]\!]]\!]_L = \{\!\{\{S, e_1\}, \{\!\{\{S, e_2\}, e_3\}\!\}\}\!\}_L$$

and applying the double Jacobi identity (2.3.12),

$$\begin{aligned} -\tau_{(123)}[\![e_1, [\![e_2, e_3]\!]]\!]_L = \tau_{(132)} \{\!\{\{S, e_2\}, \{\!\{e_3, \{S, e_1\}\}\!\}\}\!\}_L + \\ + \{\!\{e_3, \{\!\{\{S, e_1\}, \{S, e_2\}\}\!\}\}\!\}_L \end{aligned} \tag{6.4.7}$$

Next, we shall focus on $\{\!\{e_3, \{\!\{\{S, e_1\}, \{S, e_2\}\}\!\}\}\!\}_L$ and use (2.3.7):

$$\{S, \{\!\{\{S, e_1\}, e_2\}\!\}\} = \{\!\{\{S, \{S, e_1\}\}, e_2\}\!\} + \{\!\{\{S, e_1\}, \{S, e_2\}\}\!\}$$

Since $(A, \{-,-\})$ is a Loday algebra, by (6.4.5),

$$\{S, \{S, e_1\}\} = \{\{S, S\}, e_1\} - \{S, \{S, e_1\}\}$$

Thus, since $\{S, S\} = 0$ by hypothesis, we obtain the following identity:

$$\{\!\{e_3, \{\!\{\{S, e_1\}, \{S, e_2\}\}\!\}\}\!\}_L = \{\!\{e_3, \{S, \{\!\{\{S, e_1\}, e_2\}\!\}\}\}\!\}_L$$

Hence, we make the substitution in (6.4.7):

$$\begin{aligned} -\tau_{(123)}[\![e_1, [\![e_2, e_3]\!]]\!]_L &= \tau_{(132)} \{\!\{\{S, e_2\}, \{\!\{e_3, \{S, e_1\}\}\!\}\}\!\}_L \\ &\quad + \{\!\{e_3, \{S, \{\!\{\{S, e_1\}, e_2\}\!\}\}\}\!\}_L \\ &= \tau_{(132)} \{\!\{\{S, e_2\}, \{\!\{\{S, e_1\}, e_3\}\!\}^\circ\}\!\}_L + \\ &\quad + \{\!\{\{S, \{\!\{\{S, e_1\}, e_2\}\!\}\}, e_3\}\!\}_L^\circ \\ &= \tau_{(132)} \{\!\{\{S, e_2\}, [\![e_1, e_3]\!]^\circ\}\!\}_L + \{\!\{\{S, \{\!\{S, [\![e_1, e_2]\!]\}\!\}, e_3\}\!\}_L^\circ \\ &= \tau_{(132)} [\![e_2, [\![e_1, e_3]\!]^\circ]\!]_L + [\![[\![e_1, e_2]\!], e_3]\!]_L^\circ \end{aligned}$$

and we conclude (6.3.5). □

In conclusion, we proved the following

Theorem 6.4.8. *Let (A,ω,Q) be a bi-symplectic $\mathbb{N}Q$-algebra of weight 2, where A is the graded path algebra of a double quiver $\overline{P}$ of weight 2 endowed with a bi-symplectic form $\omega\in \mathrm{DR}^2_R(A)$ of weight 2 defined in §3.3.4 and a homological double derivation Q. Let B be the path algebra of the weight 0 subquiver of $\overline{P}$, and $(E,\langle -,-\rangle)$ be the pair consisting of the B-bimodule E with basis weight 1 paths in P and the symmetric non-degenerate pairing $\langle -,-\rangle := \{\!\{-,-\}\!\}_\omega|_{E\otimes E}\to B\otimes B$.*

Then the triple (A,ω,Q) determines an element $S\in A^3$ such that,

(i) *S induces a double pre-Courant–Dorfman algebra structure on $(E,\langle -,-\rangle)$ by means of*

$$\rho(e_1)(b) := \{\!\{\{S,e_1\}_\omega,b\}\!\}_\omega\,,\quad [\![e_1,e_2]\!] := \{\!\{\{S,e_1\}_\omega,e_2\}\!\}_\omega\,,$$

for all $b\in B$ and $e_1,e_2\in E$, and where $\{-,-\}_\omega = m\circ\{\!\{-,-\}\!\}_\omega$ is the associated bracket *in A.*

(ii) *The bi-symplectic $\mathbb{N}Q$-algebra (A,ω,Q) of weight 2 induces a double Courant–Dorfman algebra $(E_1,\langle -,-\rangle,\rho,[\![-,-]\!])$ over B.*

Chapter 7

Conclusions and future directions

In this thesis, double graded quivers and graded path algebras provide a natural setting to test the theory and the tools involved. In particular, we prove Theorem 3.3.40 using the explicit description of the canonical bi-symplectic form on a graded double quiver (see Proposition 3.3.34). Nevertheless, we expect that Theorem 3.3.40 can be proved for bi-symplectic tensor $\mathbb{N}$-algebras of weight 2 which do not come from quivers.

A natural question also arisen in this thesis is to prove a converse of Theorem , and let B be an associative R-algebra. As the structure of this thesis shows, motivated by Roytenberg [81], this question can be decomposed in different results, which are interesting by their own right. From now on, let R be a finite dimensional semisimple associative algebra over k, a field of characteristic 0. Firstly, to classify bi-symplectic tensor $\mathbb{N}$-algebras of weight 2 in terms of pairs $(E_1, \langle -,-\rangle)$, following Rothstein [79] (also Roytenberg [81]), we may need a notion of non-commutative curvature. We start with seeking a concept of a connection using double derivations. In the course of this work, we define these objects motivated by the definition given by J. Cuntz and D. Quillen [31] §8 (inspired in Connes' [24]):

Definition 7.0.1. Let M be any B-bimodule. A *left connection* $\nabla^l_\bullet$ on M is a right B-module map

$$\nabla^l_\bullet\colon \mathbb{D}\mathrm{er}_R B \times M \longrightarrow B \otimes M\colon \quad (X,m) \longmapsto \nabla^l_X m,$$

satisfying

$$\nabla^l_X(bm) = b\nabla^l_X(m) + X(b)m,$$

for any $b \in B$, $X \in \mathbb{D}\mathrm{er}_R B$, $m \in M$. Similarly, a *right connection* $\nabla^r_\bullet$ on M is a left B-module map

$$\nabla^r_\bullet\colon \mathbb{D}\mathrm{er}_R B \times M \longrightarrow M \otimes B\colon \quad (X,m) \longmapsto \nabla^r_X m,$$

such that

$$\nabla^r_X(mb) = \nabla^r_X(m)b + mX(b)$$

for any $b \in B$, $X \in \mathbb{D}\mathrm{er}_R B$, $m \in M$. A *connection* $\nabla_\bullet = (\nabla^l_\bullet, \nabla^r_\bullet)$ on an B-bimodule M is a pair consisting of a left connection $\nabla^l_\bullet$ and a right connection $\nabla^r_\bullet$.

Recall that being a *right B-module map* means $\nabla^l_\bullet(ma) = \nabla^l_\bullet(m)$. In addition, note that, for instance, $mX(b) = m(X'(b) \otimes X''(b)) = mX'(b) \otimes X''(b)$. However, we were not be able to define a suitable concept of non-commutative curvature which fits into our purpose. An alternative approach to construct the bi-symplectic form from the pair $(E_1, \langle -,-\rangle)$ may be given by M. Rothstein ([79] (26) - (34)), since he constructed the symplectic form in terms of the differential of a one-form which depends on the inner product and connection. In fact, we also expect some similarities between Rothstein's symplectic form and minimal coupling forms (which are important in symplectic topology and Hofer's geometry –see [47], [50], [73], [74], [91]). It can be interesting to elucidate this link and the possible implications in a non-commutative framework.

To prove the converse of Proposition 6.4.2, we start with the construction of the required function S of weight 3. Following Grützmann *et al.* [44], we have to introduce the torsion on a double Courant–Dorfman algebra $(E, \langle -,-\rangle, \rho, [\![-,-]\!])$ endowed with a connection $\nabla_\bullet$ (it is straightforward to obtain this concept from Definition 7.0.1, and Alekseev and Xu [3]). Then, inspired by [54] and [42], the *E-torsion* is defined as the weight 3 map $\mathrm{C}_\nabla \colon (E^\vee)^{\otimes 3} \to B^{\otimes 3}$ given by

$$\langle \mathrm{C}_\nabla, e_1 \otimes e_2 \otimes e_3 \rangle = \langle [\![e_1, e_2]\!]_r + \nabla^l_{\rho(e_1)} e_2 - \sigma_{(12)} \nabla^r_{\rho(e_2)} e_1, e_3 \rangle, \tag{7.0.2}$$

However, observe that the converse of Proposition 6.2.4 is based on the crucial fact that $b \in B$ can be written as $b = \{e, e'\}$ for appropriate $e, e' \in E$. To prove the converse of Proposition 6.4.6, the problem is that this trick cannot be carried out in our non-commutative setting because, by definition, $\langle -,-\rangle \colon E \times E \to B \otimes B$. So, we need an alternative approach, maybe in terms of a differential obtained from the structure of a double Courant–Dorfman algebra as Roytenberg showed in [83] (4.1). Finally, in this line of thought, it is natural to address the question of solving the *double master equation* $\{\!\{Q, Q\}\!\}_\omega = 0$. By Lemma 2.5.6, $Q = \{\!\{S, -\}\!\}_\omega$ and it translates to solve the equation $\{\!\{S, S\}\!\}_\omega = 0$. As in the setting of double graded quivers the torsion and the anchor of a double Courant–Dorfman algebra can be written explicitly, we expect that some combination of them yields a solution of the equation $\{\!\{S, S\}\!\} = 0$ and enables us to solve the double master equation and, as a consequence, we obtain examples of homological double derivations for double graded quivers.

According to the Kontsevich–Rosenberg principle, we can study the geometry of Courant algebroids on representation schemes induced by non-commutative Courant algebroids. In particular, we can determine the non-commutative Courant algebroid inducing the exact standard Courant algebroid $\mathrm{T} \oplus \mathrm{T}^*$ on the representation scheme. This may be important to establish the correct definition of exact non-commutative Courant algebroid, or transitive non-commutative Courant algebroids (recall that a Courant algebroid is *transitive* if its anchor is surjective), and to study some of their properties. In particular, we may obtain a non-commutative analogue of the Ševera class, a "curvature 3-form" classifying non-commutative exact Courant algebroids.

Furthermore, it is a classical topic in geometry that, in presence of a symmetry, a given geometrical structure may, under suitable conditions, pass to the quotient. H. Bursztyn, G. R. Cavalcanti and M. Gualtieri [17] presented a theory of reduction for *exact* Courant algebroids, which can be regarded as an "odd" analog of the usual notion of symplectic reduction due to Marsden and Weinstein. In fact, Burzstyn *et al.* specialize the reduction procedure when a moment map is involved. The idea is to translate this procedure to the non-commutative framework, using the non-commutative moment map given by Crawley-Boevey–Etingof–Ginzburg [30] §4.1 in terms of the distinguished double derivation Δ.

Finally, roughly speaking, higher Courant algebroids are analogues to Courant algebroids whose operations (the inner product and the higher-order Courant–Dorfman bracket) take place on the direct sum bundle $\mathrm{T}\,M \oplus \bigwedge^n \mathrm{T}^*\,M$, with M a fixed C^∞-manifold. In his ongoing thesis, Camilo Rengifo provides a natural context for the study of higher Courant algebroids in the framework of differential graded geometry. Therefore, we can study whether the tools developed in this thesis can be applied to define "higher non-commutative Courant algebroids" satisfying the Kontsevich–Rosenberg principle, and if we can obtain examples of these structures using double graded quivers.

Chapter 8

Conclusiones y direcciones futuras

En esta tesis, los carcajs graduados doblados y las álgebras de caminos graduadas proporcionan un contexto natural para validar la teoría y las herramientas introducidas. En particular, probamos el Teorema 3.3.40 usando la descripción explícita de la forma bi-simpléctica canónica sobre un quiver graduado doblado (ver Proposición 3.3.34). De todas formas, esperamos que el Teorema 3.3.40 se pueda probar para $\mathbb{N}$-álgebras tensoriales bi-simplécticas de peso 2 que no procedan de carcajs.

Una cuestión natural que ha surgido en esta tesis es probar el recíproco del Teorema 6.4.8. Siguiendo a Roytenberg [81] y como muestra la estructura de esta tesis, esta pregunta se puede descomponer en diferentes resultados, interesantes por derecho propio. A partir de ahora, sea R un álgebra asociativa semisimple de dimensión finita sobre k, un cuerpo de característica cero. Además, a partir de ahora consideraremos B como una R-álgebra asociativa. En primer lugar, para clasificar $\mathbb{N}$-álgebras tensoriales bi-simplécticas de peso 2 en términos de pares $(E_1, \langle -,-\rangle)$, siguiendo a Rothstein [79] (también a Roytenberg [81]), precisaremos de la noción de curvatura no conmutativa. Por ello, comenzamos buscando el concepto de conexión en términos dobles derivaciones. En el curso de este trabajo, definimos estos objetos motivados por la definición dada por J. Cuntz y D. Quillen [31] §8 (inspirada en la de Connes expuesta en [24]):

Definition 8.0.1. Sea M un B-bimódulo. Una *conexión por la izquierda* $\nabla_\bullet$ sobre M es una aplicación de B-módulos por la derecha

$$\nabla^l_\bullet \colon \mathbb{D}\mathrm{er}_R B \times M \longrightarrow B \otimes M \colon \quad (X, m) \longmapsto \nabla^l_X m,$$

satisfaciendo

$$\nabla^l_X(bm) = b\nabla^l_X(m) + X(b)m,$$

para todo $b \in B$, $X \in \mathbb{D}\mathrm{er}_R B$, $m \in M$. De manera análoga, una *conexión por la derecha* $\nabla_\bullet$ sobre M es una aplicación de B-módulos por la izquierda

$$\nabla^r_\bullet \colon \mathbb{D}\mathrm{er}_R B \times M \longrightarrow M \otimes B \colon \quad (X, m) \longmapsto \nabla^r_X m,$$

tal que

$$\nabla^r_X(mb) = \nabla^r_X(m)b + mX(b)$$

para todo $b \in B$, $X \in \mathbb{D}\mathrm{er}_R\, B$, $m \in M$. Una conexión $\nabla_\bullet = (\nabla^l_\bullet, \nabla^r_\bullet)$ sobre un B-bimódulo M es un par consistente en una conexión por la izquierda $\nabla^l_\bullet$ y una conexión por la derecha $\nabla^r_\bullet$.

Recordemos que ser una *aplicación de B-módulos por la derecha* significa $\nabla^l_\bullet(ma) = \nabla^l_\bullet(m)$. Además, observamos que, por ejemplo, $mX(b) = m(X'(b) \otimes X''(b)) = mX'(b) \otimes X''(b)$. De todas formas, no hemos sido capaces de encontrar un concepto adecuado de curvatura no conmutativa que encaje en nuestro propósito. Un enfoque alternativo para construir la forma bi-simpléctica a partir del par $(E_1, \langle -,-\rangle)$ se puede encontrar en el artículo de M. Rothstein ([79] (26) - (34)) ya que en él Rothstein construye la forma simpléctica en términos de la diferencial de una 1-forma que depende de un producto escalar y de una conexión. De hecho, esperamos encontrar similitudes entre la forma simpléctica de Rothstein y las formas minímamente acopladas (*minimal coupling forms*) que han resultado ser importantes en topología simpléctica y en la geometría de Hofer -ver [47], [50], [73], [74], [91]-). También puede ser interesante dilucidar esta conexión y las posibles implicaciones en un contexto no conmutativo.

Para probar el recíproco de la Proposición 6.4.2 empezamos con la construcción de la necesaria función S de peso 3. Sigiuiendo a Grützmann *et al.* [44] tenemos que introducir la torsión sobre un álgebra de Courant–Dorfman doble $(E, \langle -,-\rangle, \rho, [\![-,-]\!])$ dotada con una conexión $\nabla_\bullet$ (es fácil obtener este concepto a partir de la Definición 7.0.1, y Alexseeev y Xu [3]). Entonces, inspirados por [54] y [42], la E-torsión se define como la aplicación de peso 3 $\mathrm{C}_\nabla\colon (E^\vee)^{\otimes 3} \to B^{\otimes 3}$ dada por

$$\langle \mathrm{C}_\nabla, e_1 \otimes e_2 \otimes e_3\rangle = \langle [\![e_1, e_2]\!]_r + \nabla^l_{\rho(e_1)} e_2 - \sigma_{(12)} \nabla^r_{\rho(e_2)} e_1, e_3\rangle, \tag{8.0.2}$$

De todas formas, observamos que el recíproco de la Proposición 6.2.4 se basa en el hecho importante de que $b \in B$ se puede escribir como $b = \{e, e'\}$ para $e, e' \in E$ apropiados. Para probar el recíproco de la Proposición 6.4.6, el problema es que este truco no se se puede usar en nuestro contexto no conmutativo porque, por definición, $\langle -,-\rangle\colon E \times E \to B \otimes B$. Luego precisamos un enfoque alternativo, quizás en términos de una diferencial obtenida a partir de la estructura de un álgebra de Courant–Dorfman doble como Roytenberg mostró en [83] (4.1). Finalmente, en esta línea, es natural plantearse la cuestión de resolver la *ecuación maestra doble* $\{\!\{Q, Q\}\!\}_\omega = 0$. Por el Lema 2.5.6, $Q = \{\!\{S, -\}\!\}_\omega$ y esto nos lleva a querer resolver la ecuación $\{\!\{S, S\}\!\}_\omega = 0$. Como en el contexto de carcajs graduados doblados la torsión y el ancla de un álgebra de Courant–Dorfman doble se puede escribir en términos muy explícitos, esperamos que alguna combinación de estos objetos nos proporcione una solución de la ecuación $\{\!\{S, S\}\!\} = 0$ y, como consecuencia, obtener ejemplos de derivaciones dobles homológicas para carcajs dobles graduados.

De acuerdo con el principio de Kontsevich–Rosenberg, podemos estudiar la geometría de los algebroides de Courant sobre los esquemas de representaciones inducidos por los algebroides de Courant no conmutativos. En particular, podemos determinar el algebroide de Courant no conmutativo que induce el algebroide de Courant estándar $\mathrm{T} \oplus \mathrm{T}^*$ sobre los esquemas de representaciones. Esto puede ser importante para establecer la definición correcta de algebroides de Courant no conmutativos exactos o algebroides de Courant no conmutativos transitivos (recordemos que un algebroide de Courant es *transitivo* si su ancla es sobreyectiva), y para estudiar algunas de sus propiedades. En particular, podemos obtener un análogo no conmutativo de la clase de Ševera, una “3-forma de curvatura” que clasifique los algebroides de Courant no conmutativos exactos.

Además, es un tema clásico en geometría que, en presencia de una simetría, una estructura geométrica dada, bajo condiciones adecuadas, puede descender al cociente. H. Bursztyn, G. R. Cavalcanti and M. Gualtieri [17] presentaron una teoría de reducción para algebroides de Courant *exactos*, que se puede ver como un análogo “extraño” de la noción usual de reducción simpléctica debida a Marsden y a Weinstein. De hecho, Burzstyn *et al.* especializaron el procedimiento de reducción en presencia de una aplicación momento. La idea es traducir este procedimiento al contexto no conmutativo, usando la aplicación momento no conmutativa dada por Crawley-Boevey–Etingof–Ginzburg [30] §4.1 en términos de la doble derivación distinguida Δ.

Finalmente, en téerminos poco rigurosos, podemos decir que los algebroides de Courant de orden superior son análogos a los algebroides de Courant cuyas operaciones (el producto escalar y el corchete de Courant–Dorfman de orden superior) tienen lugar sobre el fibrado $\mathrm{T}\,M \oplus \bigwedge^n \mathrm{T}^*\,M$, con M una C^∞-variedad. En su tesis en curso, Camilo Rengifo construye un contexto natural para el estudio de los algebroides de Courant de orden superior usando geometría diferencial graduada. Por tanto, podemos estudiar si las herramientas desarrolladas en esta tesis pueden ser útiles para definir ”algebroides de Courant de orden superior no conmutativos” satisfaciendo el principio de Kontsevich–Rosenberg, y si se pueden obtener ejemplos de estas estructuras usando carcajs graduados dobles.

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